MATHEMATICS IN THE ARMY EDUCATION PROGRAM

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1. Introduction. The purpose of this paper is to present the facts with regard to mathematics in the Army Education Program and some of the implications which can be derived from these facts. In particular, I shall consider the question: How much demand for mathematics can be expected from the veteran?

Mathematics has been one of the most popular subjects in the curriculum of the Army Education Program. In order that the reader may understand the full significance of this popularity, some of the pertinent points of the Army Education Program should be explained. This program was begun prior to our entry into World War II, and had as its purpose the furnishing to the serviceman of educational opportunities to fit his needs or desires, or both. In particular, the subjects offered were selected (1) to enhance the value of the individual to the Army, (2) to prevent a complete interruption of the serviceman's education, (3) to satisfy intellectual interests, and (4) to prepare the serviceman for his return to civilian life. The method used at first was the furnishing of correspondence courses and lesson service to individuals upon their request. The program was gradually enlarged to include self-study courses, off-duty and on-duty classes, and university extension courses from some seventy-five colleges and universities. The curriculum begins with literacy training, and includes, in addition to the academic courses, a wide variety of vocational subjects. The United States Armed Forces Institute at Madison, Wisconsin, is the principal installation in the implementation of this program. The Institute is in reality a large correspondence school, a testing agency, and a supply depot, and is operated jointly by the War and Navy Departments.

There is one feature of this program which should be emphasized, namely, that participation has always been voluntary on the part of the soldier. It is this fact which makes the popularity which courses in mathematics enjoyed especially startling. When a man becomes a member of the student body of the United States Armed Forces Institute, he does so simply because he wants to learn something that he does not know, and for this reason alone. There is no social prestige, no football team, no paternal alma mater involved in his decision. Of course, men are encouraged to take advantage of the opportunities offered and are given counsel and guidance in the selection of subjects. About ten percent of the personnel participate in the program, and this is encouraging because the surroundings, environment, and facilities are not particularly conducive to study. There are no well-lighted comfortable classrooms, no warm inviting study halls with instructional help and ample library facilities. During hostilities the conditions were particularly difficult, and official duties left time for little else. During the period of demobilization there was much more time available, but the facilities for study were largely improvised and the environment far from ideal.

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2. Courses offered in the program. As stated above, the curriculum runs the gamut of mathematics. It begins with literacy training in which the most elementary principles of arithmetic are taught. In the high school category such books as *Basic Mathematics* by Betz, *Algebra* by Wells and Hart, and *Trigonometry* by Curtiss and Moulton are the texts. Hart's *College Algebra*, three standard trigonometries, Love's *Analytic Geometry*, Granville, Smith and Longley's *Calculus*, and Murray's *Differential Equations* are examples of the college texts. These books show what type of subject matter the servicemen have been choosing; certainly they are texts designed for learning and not for amusement. The above courses are offered by the United States Armed Forces Institute itself.

Further courses in mathematics are available to the serviceman through the extension divisions of the universities and colleges under contract with the Institute. However, in this paper only Institute courses are considered.

It should be mentioned here that most of the books used in the Army Education Program were bound in paper for reasons of economy in purchase price and shipping space. The majority of the texts are identical in content with the high school or college edition. Some have been revised by the insertion of additional explanation, exercises, and reviews, so as to enable the student to understand and absorb the subject matter without the aid of an instructor. The purpose is a "self-teaching text." It was the Army's plan to provide self-teaching texts for all subjects amenable to such treatment; however, the project was curtailed after V-J Day. Consequently, of the total number of texts in the curriculum of the United States Armed Forces Institute, only a few are "self-teaching."

3. Popularity of courses. In considering the popularity of books and courses, the wide variety of offerings is important. The curriculum included more than 400 courses when these figures were compiled. There were courses in everything from "Learning to Read" to "Shakespeare," from "Pork Production" to "Art Through the Ages."

Of the twenty most popular books, seven are mathematics, all high school texts, and four others are bookkeeping and accounting. The most popular book of all is *Auto-Mechanics*; second is *Bookkeeping and Accounting;* third is *A First Course in Algebra, Part I;* and fourth is *A First Course in Algebra, Part II*. The only college level books among the twenty most popular are *Principles of Accounting* and *Modern Electric and Gas Refrigeration*. All of the books except the last two are "self-teaching."

Of the twenty most popular high school correspondence courses, five are mathematics. The most popular course is Beginning Algebra; second is Review Arithmetic; and third, Bookkeeping. Trigonometry is ninth on the list. Twenty percent of all those enrolled in high school correspondence courses are taking mathematics. The second most popular field is bookkeeping, which has fifteen percent of the enrollees. Of those studying mathematics, six percent are taking arithmatic, five percent algebra, and four percent plane geometry.

Of the twenty most popular college correspondence courses, four are mathematics. The most popular course is College Algebra and Trigonometry; second

Introduction to Accounting; and third, English Composition. Differential Calculus ranks tenth; Differential Equations is twenty-eighth; and Integral Calculus, thirty-sixth. Twenty-four percent of all those enrolled in college correspondence courses are taking mathematics. The second most popular field is accounting, which has twelve percent of the enrollees. The mathematics course with the highest percentage enrollment is College Algebra and Trigonometry; nine percent of all the enrollees are taking this subject. Differential Calculus enjoys a surprising demand; five percent of those enrolled are taking this course.

A poll taken in July of this year shows that eighteen percent of the men in the Army have signed up for USAFI correspondence courses. Over 75,000 men have enrolled for mathematics courses at the Institute. Of the courses currently being distributed, about nine percent are mathematics courses. Sixteen percent of the active enrollees, as of July 1, 1946, were taking mathematics. Of total completions, as of July 1, 1946, approximately eight percent were in mathematics courses.

4. Reasons for the popularity of mathematics. Now, what do all these figures mean? The first question is why such a horrible subject as mathematics should be so popular? Why should so many men of their own free will and accord elect to study algebra or the calculus? One reason is certainly that military operations require quantitative thinking. The old maxim of Nathan Bedford Forrest that the way to win a battle is "to get there fustes with the mostes men" is still pertinent with slight revision, say, "to get there fustes with the mostes power." Every part of this statement involves quantities. Power means men, ammunition, guns, equipment, vehicles, fuel, and so on. Time and distance and speed are involved. In fact, a military operation involves figures from start to finish. Every man in the operation works against these figures, and there are very few participants who are not acutely conscious of their part in a colossal mathematical problem. It is undoubtedly true that many men were so continuously bombarded with figures and calculations that they resolved to learn something about arithmetic and algebra in order to better understand the task before them. Others saw this lack of knowledge hampering them in the job they were doing or preventing them from being promoted. Certainly the military environment encouraged the study of mathematics, and accounts for some of the popularity which the subject enjoys.

But this is not the predominant factor. Many of the men who took these courses enrolled at a time when the fighting was nearing its end and when they were looking forward not to spending much more time in the service, but to getting out of the Army and going back to school or to work. It is true that their recent years of Army life had emphasized quantitative thinking and this had its effect on their choice of subjects. On the other hand, their post-war career was uppermost in their minds. Many men were planning to go back to school, and they realized that they did not have the mathematical background which they needed. They were more interested in what was valuable for their future than in what credits they needed to get a diploma.

5. Decrease in emphasis on mathematics. This brings up a point which is very important to every mathematician. It is the decreasing mathematical content of the high school and college curricula during the past twenty or thirty years. Twenty-five years ago, every student was required to take two years of good, sound algebra in order to graduate; in addition, two more years of mathematics were offered as electives and had many subscribers. Moreover, two years of mathematics were required for an A.B. degree in many colleges. The present situation is a far cry from that, even in the most conservative institutions. Mathematics has gradually been removed from the various curricula until there is very little left that is useful or even recognizable. Many of the courses which are called mathematics are a disgrace to the name. They are designed for amusement, and anything which might be thought-provoking is carefully avoided.

It is difficult to understand why such a purge could take place in the face of the trend of modern civilization. The old arguments for studying mathematics were that it developed logical processes which were useful in practically every walk of life and that it was indispensable for many trades and professions. These arguments are certainly just as valid today as they ever were. Furthermore, the present emphasis on science and technology makes a knowledge of some mathematics necessary if one is to understand, even superficially, the foundation of modern progress.

Notwithstanding, arithmetic and algebra have been placed in the high school museum, and college courses in mathematics have been retained for those few who want to study science or engineering. The only way to put the subject back in its proper place in college is to replace it in the high school curriculum. It was removed from the high schools because it was reputed to be dry, difficult, and impractical. As a matter of fact, this reputation was probably deserved in many cases, and no doubt the very instructors who were handling the subject admitted just that. Certainly mathematics is difficult for many students; it cannot be made easy and still be mathematics. On the other hand, there is no reason why it should be dry and impractical if it is presented properly. Proper presentation means suitable textbooks and capable teachers, and these are forces which can stem the present tide, a tide which will lead inevitably to the virtual extermination of mathematics as a universal subject.

In the years before the war, professors of mathematics in colleges, and especially in engineering schools, saw the effect of the diminishing mathematical knowledge of freshmen. It was obvious after two or three weeks that a high percentage of the entering students did not know enough arithmetic, much less algebra, to pursue the course in college algebra. There were only two solutions: (1) to eliminate those who could not make the grade, or (2) to go back and teach them arithmetic and high school algebra. Of course, the second solution was preferable for very obvious reasons: the number was too large to eliminate, such action would result in an indefensible waste of talent, and many boys who were capable but just poorly trained would be done a rank injustice. By establishing a course in arithmetic and high school algebra, many colleges were able to sal-

vage those who had the ability and determination to do the work; however, the students involved lost a semester which had to be made up at a later date. Moreover, much college faculty time was devoted to teaching high school mathematics. If such a situation is allowed to continue, one of two things must happen: either the mathematics once belonging to the secondary school category must be reclassified as college mathematics and the level lowered, or else the colleges must extend their courses beyond the present four years. In any event, college professors can expect more and more of their time to be devoted to teaching what they now consider high school mathematics.

6. A correction of this situation. This situation can be corrected, and the way mathematics has fared in the Army Education Program is an indication of why it should be corrected and it shows that now is the time. Many thousands of GI's who, as high school students, were allowed to skip over mathematics without learning it, registered for arithmetic, algebra, and geometry courses in the Army. They resent having been allowed to omit as superfluous something which later proved to be essential. They know now that it is inadvisable to allow a high school student to take only those subjects which he selects, because they are easy or because he likes them. They saw themselves faced with problems for which they had not been properly prepared, even though they could and should have been prepared in the normal course of events. As men, they realized that their judgments as boys were not sufficiently developed to rely upon in vital areas. These men believe in mathematics and the type of mathematics which has content. They are allies in putting the subject back in its place in the schools and colleges.

This means better teachers and better texts, and it is up to the mathematicians to develop them. It also means educating the public to the advantages of a knowledge of mathematics and to the disadvantages of an ignorance of the subject. The public is ready to listen to this explanation because it is now infiltrated with millions of men who saw the necessity for such knowledge during the war and with thousands who did something about it by studying on their own time and of their own volition.

But replacing arithmetic and algebra in the high school curriculum is only part of the job. Many of those men who became interested in mathematics while they were in the Army will never become full-time students again. They left school years ago, and circumstances do not permit them to return at this late date. However, they know what mathematics has to offer and they would welcome an opportunity to pursue the subject further. But the opportunity must be real; it should be an invitation. Such men are not likely to become engrossed in the contents of an algebra book written for the adolescent, geared to average student indifference, and dependent upon instructional explanation. They will respond to texts designed for self-teaching and directed to the adult in explanation, in illustrations, and in speed. This does not mean watered-down subject matter designed to be amusing. In fact, it means just the opposite; it means

real, down-to-earth mathematics designed to be interesting and practical. The text should be challenging and hence should be built around the average adult's experiences. Unless such books are made easily available and those who are interested encouraged to make use of them, many prospective customers will be lost.

The aim is to get mathematics into the adult education programs now gaining impetus in this country. This will require in addition to suitable books a concerted effort toward informing the public. The author will probably be criticized for calling for an advertising campaign, but mathematics is an article which is worth advertising and which can be sold. When one considers how many customers there were among the GI's without particular effort on anyone's part, one naturally wonders what could be done intentionally, knowing that startling results were obtained incidentally. All of the participants in the Army Education Program are adults and they were using books designed for high school and college students. How many more participants would there have been if there had been sufficient time to develop more suitable texts!

7. Continuation of the Army Education Program. Finally, it should be added that the Army Education Program will continue and that mathematics will be given as much opportunity in the future as in the past. The program will follow about the same principles as heretofore. Due to the decreased size of the Army, the numbers involved will be much smaller. Also, now that the draft has been suspended and emphasis is on voluntary enlistments, the educational level has dropped. The demand is centering on high school and vocational courses, and many of the college courses are no longer sufficiently popular to be retained in the curriculum. In fact, at the last meeting of the War-Navy Committee on the United States Armed Forces Institute it was decided to drop from the Institute offerings all courses above the freshman college level except those of particular value to the military. It is interesting to note that in this streamlining, the only mathematics course to be deleted was Differential Equations; even Differential and Integral Calculus were retained because they are still, in the reduced Army, very much in demand. Of course, every type of college course still remains available to Army personnel through college extension courses. As far as mathematics is concerned, the Army will continue to offer a wide choice of subjects directly from the Institute and in classes, and there is every reason to expect that the GI will continue to study everything from arithmetic through the calculus.

ESTIMATING ELECTROSTATIC CAPACITY*

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1. Introduction. To find the electrostatic capacity of a given solid is a clear-cut, relatively simple problem of applied mathematics.

It is indeed a clear-cut problem. In order to solve it, we have to find first an integral of a differential equation under given boundary conditions (a function satisfying Laplace's equation outside the solid, that reduces to 1 on the boundary of the solid and to 0 at infinity). Then, we have to pick out a coefficient from the expansion of the integral we have found (the coefficient of $1/r$ in the expansion into spherical harmonics valid at sufficiently great distance from the origin). We can take the value of this coefficient as the definition of the electrostatic capacity of the solid if we are pure mathematicians.

To find the electrostatic capacity so defined, though difficult enough, is a problem that is simple in comparison with several others in applied mathematics. There are at least a few cases (ellipsoid, lens, spindle, anchor-ring) in which an exact solution can be found. That is, in these cases we can express the capacity as an integral or an infinite series whose numerical evaluation may still be quite difficult. In a very few cases, however, the expression of the capacity is simple, and it is quite simple in case of the sphere whose capacity is numerically equal to its radius. Yet for many familiar solids (as cube, right circular cylinder, and so on) we do not know the capacity, and have little hope of obtaining an exact expression for it.†

Yet "even when analysis fails to give a solution in the mathematical sense, we need not be altogether in the dark as to the magnitudes of the quantities with which we are dealing."‡ We may still obtain approximate formulas or definite bounds. Helped by a grant from the Office of Naval Research and a few collaborators, Mr. Szegö and the author undertook to explore systematically bounds for, and approximations to, the electrostatic capacity. We regard the evaluation of capacities as a sort of test case, hoping that some of the methods may have a wider scope and be used also for related problems of applied mathematics. Of the various results that we found some will be outlined in what follows. The methods of proof, however, cannot even be suggested in this short communication.§

2. Volume-radius. Capacity has an intuitive meaning for the physicist, but

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† It was related by Jacobi and Kirchhoff that Dirichlet, shortly before his death, solved the problem of distribution of electricity on a rectangular parallelepiped. This solution appears to be lost.

‡ Lord Rayleigh, The Theory of Sound, 2nd ed., vol. 2, (1896) p. 180.

§ All new facts outlined in the following result from the joint work of Mr. Szegö and the present author, undertaken in September 1946, as a sequel to the work presented in the paper, G. Pólya and G. Szegö, Inequalities for the capacity of a condenser, American Journal of Mathematics, vol. 67, (1945) pp. 1–32.

another interpretation of the same quantity may appeal more to people less familiar with electrical experiments. We imagine a body embedded in a uniform infinite medium whose temperature at infinity is 0. If the surface of the solid is kept at a constant temperature (as the skin of a man or a cat approximately is) a steady flow of heat will pass from the body into the medium. Let 1 be the constant temperature of the boundary. Then the amount of outgoing heat per unit time is called the *thermal conductance* and this quantity is, except for a constant factor depending on the nature of the medium, equal to the electrostatic capacity of the embedded body.

Now we can all observe what a cat does when he prepares himself for sleeping through a cold night: he pulls in his legs, curls up, and, in short, makes his body as spherical as possible. He does so, obviously, in order to minimize the thermal conductance or, what is the same thing, his capacity. Thus, he seems to have a sort of knowledge of the following general theorem: *Of all solids with a given volume, the sphere has the minimum capacity.*

This theorem can actually be proved,* and since the capacity of a sphere is known, it can be restated as follows:

THEOREM. *The capacity of a solid is never less than the radius of a sphere with equal volume.*

Let C denote the capacity and V the volume of the solid. Then the fact just stated can be expressed by the inequality

$$C > \left(\frac{3V}{4\pi}\right)^{1/3}, \tag{1}$$

which is valid unless the solid is a sphere. The quantity on the right side of (1) may be called the *volume-radius* of the solid having the volume V. Thus, for any solid, the ratio of the capacity to the volume-radius is greater than 1. Examples show that this ratio can be arbitrarily large. The volume-radius yields a simple lower bound but no upper bound for the capacity.

3. Surface-radius. If the area of the surface of a solid is S, the radius of a sphere with the same surface area is $(S/4\pi)^{1/2}$, and we call this quantity the *surface-radius*. The ratio of capacity to surface-radius can attain any given positive value if the solid is appropriately chosen. In other words, this ratio varies between 0 and ∞. Thus, the surface-radius yields no usable upper or lower bound for the capacity.

This situation changes, however, if we restrict ourselves to the consideration of *convex* solids. Then the ratio in question has a positive lower bound, and we conjecture that *of all convex bodies, with a given surface area, the circular disk has*

* The theorem was discovered by H. Poincaré, Figures d'équilibre d'une masse fluide, Paris, 1903, pp. 17–22, yet without a complete proof. The first complete proof was given by G. Szegö, Über einige Extremalaufgaben der Potentialtheorie, Mathematische Zeitschrift, vol. 31 (1930) pp. 583–593.

the minimum capacity. This rather plausible conjecture is expressed by the inequality

$$C > \frac{8^{1/2}}{\pi}\left(\frac{S}{4\pi}\right)^{1/2}, \tag{2}$$

which we expect to be valid unless the convex body becomes a circular disk. In case of the disk, the two members of (2) become equal.

Our conjecture is not without a certain basis. We have verified it in various special cases and, although we did not prove (2), we did prove the weaker inequality

$$C > \frac{2}{\pi}\left(\frac{S}{4\pi}\right)^{1/2}. \tag{2'}$$

4. Mean radius. Here we restrict ourselves from the outset to the consideration of convex solids. The distance between two parallel tangent planes (or, more generally, "supporting" planes) is called the width of the solid in the direction perpendicular to those planes. Integrating over all directions (over the uniformily covered surface of the unit sphere), we find the *average width* of the solid, and we call one half of the average width the *mean radius.* As we can see immediately from a formula due to Minkowski, the mean radius of a "smooth" convex solid is also equal to $M/4\pi$, where M denotes the surface integral of the mean curvature extended over the whole boundary of the solid. The mean radius can easily be computed for any convex polyhedron. For instance, it is $3a/4$ for a cube with edge a.

In general we have the following inequalities between the three radii here defined:

$$\left(\frac{3V}{4\pi}\right)^{1/3} < \left(\frac{S}{4\pi}\right)^{1/2} < \frac{M}{4\pi} \tag{3}$$

unless the convex solid is a sphere, in which case all three radii become equal. This is a restatement of classical results. The first of these inequalities expresses the "isoperimetric" property of the sphere; the second is due to Minkowski.*

We come now to the following result† which is not so easy to foresee intuitively as inequalities (1) or (2):

THEOREM. *The capacity of a convex solid is never superior to the capacity of a prolate ellipsoid whose major semi-axis is the mean radius of the solid and whose minor semi-axis is the surface-radius.*

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* See H. Minkowski, Gesammelte Abhandlungen (1911) vol. 2, pp. 103–279, or W. Blaschke, Kreis und Kugel (1916), or T. Bonnesen and W. Fenchel, Theorie der konvexen Körper(1934), expecially p. 66 and p. 110 of the last quoted work.

† G. Szegö, Über einige neue Extremaleigenschaften der Kugel, Mathematische Zeitschrift, vol. 33 (1931) pp. 419–425. The formulation here given is new.

This states that, unless the solid is a sphere,

$$C < \frac{M}{4\pi} \frac{2\epsilon}{\log \frac{1+\epsilon}{1-\epsilon}}, \tag{4}$$

where the positive quantity (eccentricity) ϵ is defined by the equation

$$\epsilon^2 = 1 - \frac{4\pi S}{M^2}.$$

From (4), we can derive the weaker inequality

$$C < \frac{\frac{M}{4\pi} + 2\left(\frac{S}{4\pi}\right)^{1/2}}{3} \tag{4'}$$

and the still weaker relation

$$C < \frac{M}{4\pi}. \tag{4''}$$

Joining the inequalities (1) and (4''), we obtain

$$\left(\frac{3V}{4\pi}\right)^{1/3} < C < \frac{M}{4\pi} \tag{5}$$

unless the convex solid is a sphere. A comparison of (3) and (5) reveals a remarkable parallelism between the surface-radius and the capacity.*

5. Conformal radius. The following consideration is restricted to solids of revolution. A plane passing through the axis of revolution intersects the solid in a closed curve which we call the *meridian section*. Thus, the meridian section is a plane curve having an axis of symmetry.

We map the infinite part of the plane exterior to the meridian section onto the exterior of a circle of radius $\bar{r}$, so that the points at infinity correspond to each other and the linear magnification at infinity is 1. As is well known, the quantity $\bar{r}$ is uniquely determined. We call $\bar{r}$ the *conformal radius* of the solid of revolution.

The ratio of capacity to conformal radius can be arbitrarily small but it certainly cannot be arbitrarily large. We conjecture that

$$C < \frac{4}{\pi}\bar{r} \tag{6}$$

* For further remarks about this see the joint paper by G. Pólya and G. Szegö, quoted above, On the capacity of a condenser, Bulletin of the American Mathematical Society, vol. 51 (1945) pp. 325–350.

unless the solid of revolution becomes a circular disk and its meridian section a straight line-segment. We cannot prove this conjecture yet, but we did prove that

$$C < 1.2754\bar{r} \tag{6'}$$

which is not so bad numerically since

$$\frac{4}{\pi} = 1.2732 \cdots .$$

A simply closed plane curve which has two axes of symmetry perpendicular to each other can be rotated about each of these axes and thereby generates two solids of revolution which are, in general, different. Let C and C' denote the capacities of these two solids, respectively, and let $\bar{r}$ stand again for their common conformal radius. We conjecture that

$$C + C' < 2\bar{r} \tag{7}$$

unless the meridian section becomes a circle and the two solids of revolution become equal spheres, in which case $C = C' = \bar{r}$. Again we did not prove (7) but we did prove that

$$C + C' < 2.082\bar{r}. \tag{7'}$$

6. Numerical applications. The foregoing results are not only of great generality and simplicity, but also are readily applicable. Take any convex solid. We can compute V, S, and M incomparably more easily than C. For solids of revolution we should also compute $\bar{r}$, a quantity which is of more intricate nature than V, S, or M, but still more accessible than C. Applying the bounds considered in the foregoing discussion, we often can find an acceptable estimate for C. Let us consider just two examples.

EXAMPLE 1. For a cube with edge a, we find

$$.62211a < C < .71055a.$$

The upper estimate is based on (4). The lower estimate is based on the hypothetical (2), but the well established (1) gives an estimate not much worse, namely, $.62033a$.

EXAMPLE 2. Consider the right circular cylinder obtained by rotating a square with side a about the line joining the midpoints of two opposite sides. We find

$$.57236a < C < .59017a.$$

The lower estimate is based on (1), which in this case is better than (2). The upper estimate is based on the hypothetical (7). Observe that in the present case $C = C'$.

We see that in case of the cube we can derive an approximate value for the capacity differing by less than 7% from the unknown correct value. In case of

the cylinder just considered, the difference between obtainable approximation and unknown correct value is only about 1.5% of the latter.

I may venture the opinion that it would be desirable to treat other physical quantities in the same manner as the capacity was treated in the foregoing discussion. We should establish general bounds readily applicable to special cases and yielding estimates with errors of only a few percents. This is not easy, and the outlined theory needs considerable further work even in the relatively simple case of the electrostatic capacity here discussed.

A TRIGONOMETRIC INFINITE PRODUCT

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1. Introduction. Seventy years ago, Dobinski [1] gave a curious trigonometric formula which we shall discuss. The Dobinski formula is

$$(\tan x)(\tan 2x)^{1/2}(\tan 4x)^{1/4}(\tan 8x)^{1/8}\cdots = 4\sin^2 x??. \tag{1}$$

Assuming that the individual factors are unambiguously defined, the left side of (1) is an infinite product, and the meaning of (1) is the following: if $P_n(x)$ is the product of the first n factors of (1), namely,

$$P_n(x) = (\tan x)(\tan 2x)^{1/2}(\tan 4x)^{1/4}\cdots(\tan 2^n x)^{1/2^n}, \tag{2}$$

then

$$\lim_{n\to\infty} P_n(x) = 4\sin^2 x. \tag{3}$$

The two question marks in (1) are intended to question the manner in which the factors are defined and the validity of the formula.

Let A denote the set of real numbers x not of the form $h\pi/2^k$, where h is an odd integer and k is a positive integer. When x is in A, no one of the arguments $x, 2x, 4x, 8x, \cdots$ is an odd multiple of $\pi/2$ and the tangents in (1) all exist. Dobinski gave two "proofs" that (1) holds for each x in A. It appears that he gave two proofs because he considered each to be interesting, and not because he considered each to be defective and hence needed the support of the other. We shall present his first "proof," with appropriate remarks required to clarify the formula.

2. Validity of the formula. The "proof" is based on the identity

$$\tan\theta = \frac{2\sin^2\theta}{\sin 2\theta}.$$

Taking θ to be successively $x, 2x, 4x, \cdots, 2^n x$, we obtain

$$\tan x = \frac{2(\sin x)^2}{(\sin 2x)}$$

$$(\tan 2x)^{1/2} = \frac{2^{1/2}(\sin 2x)}{(\sin 4x)^{1/2}}$$

$$(\tan 4x)^{1/4} = \frac{2^{1/4}(\sin 4x)^{1/2}}{(\sin 8x)^{1/4}}$$

. . . . . .

$$(\tan 2^n x)^{2^{-n}} = \frac{2^{2^{-n}}(\sin 2^n x)2^{-n+1}}{(\sin 2^{n+1}x)^{2^{-n}}}.$$

Multiplication and simplification give

$$P_n(x) = \frac{[2^{1+1/2+1/4+\cdots+2^{-n}} \sin^2 x]}{(\sin 2^{n+1}x)^{2^{-n}}}. \tag{4}$$

Since the quantity in brackets converges to $4 \sin^2 x$ as $n \to \infty$, (1) holds if and only if*

$$\lim_{n\to\infty} (\sin 2^{n+1}x)^{2^{-n}} = 1. \tag{5}$$

At this point we pause to look at (1) more critically. It is easy to show that for each x in A not of the form $m\pi$, m an integer, some of the arguments $2x, 4x, 8x, \cdots$ lie in the second or fourth quadrants and hence have negative tangents. Thus (1) involves roots of negative numbers. As we shall see in Section 3, (1) will hold only in trivial cases if we define $q^{1/r}$, when $q<0$ and r is an even positive integer, in the simplest and familiar unambiguous way by the formula

$$q^{1/r} = \exp[(1/r)(\log|q| + \pi i)]. \tag{6}$$

Hence we shall see what can be done by allowing the symbol $q^{1/p}$ to stand for any one of the pth roots of q which we decide to select.

Choose a value for each of the factors and consider the sequence of complex numbers $P_n(x)$. For $P_n(x)$ to converge to $4 \sin^2 x$ it is necessary and sufficient that

$$\lim_{n\to\infty} \operatorname{amp} P_n(x) = 2m\pi, \qquad m \text{ an integer}, \tag{7}$$

$$\lim_{n\to\infty} |P_n(x)| = 4 \sin^2 x. \tag{8}$$

We first consider (7). Suppose $\operatorname{amp}(\tan kx)^{1/k}$, $k=2, 4, 8, \cdots$, is confined to the interval $-\pi<\theta\leqq\pi$, and let $\operatorname{amp} P_n(x)$ have the value,

$$\operatorname{amp} P_n(x) = \operatorname{amp} \tan x + \operatorname{amp} (\tan 2x)^{1/2} + \cdots + \operatorname{amp} (\tan 2^n x)^{2^{-n}}.$$

Then (7) is equivalent to

* Here Dobinski's argument breaks down, as his "proof" of (5) involves an unjustifiable interchange of two limit processes.

$$\sum_{n=0}^{\infty} \text{amp } (\tan 2^n x)^{2^{-n}} = 2m\pi. \tag{9}$$

That the terms of this series can be chosen to make the series converge to the proper value is evident from the fact that the amplitudes of $(\tan 2^n x)^{2^{-n}}$ are equally spaced at intervals of $2\pi/2^n$. Thus in particular we can make successive choices so that we always have

$$0 \leqq \text{amp } P_n(x) \leqq \frac{2\pi}{2^n},$$

thereby satisfying (9) for $m=0$.

In discussing condition (8) we shall confine our attention to those values of x in A which lie in the interval $0<x<\pi/2$. The extension of our results to other values of x in A is a simple matter. Putting absolute values in the formulas leading to (5), we see that (8) is equivalent to

$$\lim_{n\to\infty} |\sin 2^{n+1}x|^{2^{-n}} = 1,$$

or

$$\lim_{n\to\infty} 2^{-n} \log_2 |\sin 2^{n+1}x| = 0. \tag{10}$$

Any number x in the interval $0<x<\pi/2$ can be expressed in the dyadic form

$$x = \tfrac{1}{2}\pi(.d_1 d_2 \cdots d_n \cdots), \tag{11}$$

each d_i being 0 or 1. The values of x for which there are only a finite number of 1's, or of 0's, are precisely those which were excluded from the set A, and so we may assume that the expression for x contains an infinite number of both 0's and 1's.

We have

$$2^{n+1}x = \pi(d_1 d_2 \cdots d_n . d_{n+1} d_{n+2} \cdots),$$

and

$$|\sin 2^{n+1}x| = \sin \pi(.d_{n+1} d_{n+2} \cdots).$$

CASE I. $d_{n+1}=0$. For this case,

$$0 < \pi(.d_{n+1} d_{n+2} \cdots) < \frac{\pi}{2}.$$

Now when $0<\theta<\pi/2$, it can easily be proved that

$$\frac{2}{\pi} < \frac{\sin\theta}{\theta} < 1.$$

Hence

$$|\sin 2^{n+1}x| = A_n(x)\pi(.0 d_{n+2} d_{n+3} \cdots),$$

where $A_n(x)$ is a number between $2/\pi$ and 1. Taking the logarithm of each member gives

$$\log_2 \left| \sin 2^{n+1}x \right| = \log_2 A_n(x)\pi - z_n - \phi_n,$$

where z_n is the number of 0's following d_n before the first 1 makes its appearance, and ϕ_n is a number between 0 and 1. Thus

$$2^{-n} \log_2 \left| \sin 2^{n+1}x \right| = 2^{-n}(\log_2 A_n(x)\pi - \phi_n) - 2^{-n}z_n.$$

Now

$$\lim_{n\to\infty} 2^{-n}(\log_2 A_n(x)\pi - \phi_n) = 0,$$

and so (10) cannot hold unless

$$\lim_{n\to\infty} 2^{-n}z_n = 0. \tag{12}$$

CASE II. $d_{n+1}=1$. In this case,

$$\begin{aligned} \sin \pi(.1d_{n+2}d_{n+3}\cdots) &= \sin \pi(1 - .0e_{n+2}e_{n+3}\cdots) \\ &= \sin \pi(.0e_{n+2}e_{n+3}\cdots) \end{aligned}$$

where $e_k+d_k=1$, $k=n+2, n+3, \cdots$. The argument of Case I can now be applied to show that (10) cannot hold unless

$$\lim_{n\to\infty} 2^{-n}w_n = 0, \tag{13}$$

where w_n is the number of 1's following d_n before the first 0 makes its appearance.

Thus (12) and (13) are necessary conditions for (10). Taken together they are also a sufficient condition, for each value of n comes under one of the two cases.

To sum up, we can say that Dobinski's formula (1) holds for a real number x provided that

A. x does not have the property that $2x/\pi$, expressed as a dyadic fraction, has such phenomenally long blocks of 0's or 1's that (12) or (13) fail to hold.

B. The values of $(\tan 2^n x)^{2^{-n}}$ are suitably chosen complex numbers.

3. Failure of (1). Each subinterval of the interval $0<x<\pi/2$ contains points of the form $h\pi/2^k$ where h is an odd integer and k is a positive integer; for such values of x, at least one of the tangents in (1) is undefined and the formula fails to hold. The analysis of the preceding section shows that each subinterval contains additional points (some of the points of the set A defined above) for which (1) fails to hold. In fact, for each subinterval there is an integer k such that the subinterval contains points

$$x = \tfrac{1}{2}\pi(.d_1d_2\cdots d_kd_{k+1}\cdots)$$

where the numbers d_k, d_{k+1}, $\cdots$ form a sequence consisting in order of $n_1=2^k$ zeros, $n_2=2^{k+n_1}$ ones, $n_3=2^{k+n_1+n_2}$ zeros, and so on. For each such x, (12) and (13) fail to hold, and hence also (8) and (1) fail to hold.

Suppose now that x does not have the form $h\pi/2^k$ where h and k are integers,

so that the tangents in (1) all exist and are all different from zero. When the roots in (1) are determined unambiguously by use of (6), the series of amplitudes in the left member of (9) becomes

$$\pi\left(\alpha_1+\frac{\alpha_2}{2}+\frac{\alpha_3}{2^2}+\frac{\alpha_4}{2^4}+\cdots\right) \tag{14}$$

where each α is 0 or 1 and the α's are neither all 0 nor all 1. The series (14) necessarily converges to an amplitude between 0 and 2π. Accordingly (7) and Dobinski's formula (1) fail to hold, except in trivial cases, when the roots are unambiguously determined in the simplest and usual way.

4. Application of Borel's theorem on dyadic expansions. Borel [2] showed that for each x in $0<x<1$, except for those in a set having Lebesgue measure zero, the dyadic expansion of x, namely,

$$x = .d_1d_2d_3\cdots$$

has the following property: if p_n is the number of zeros in the first n places, then

$$\lim_{n\to\infty}\frac{p_n}{n}=\frac{1}{2}. \tag{15}$$

If (15) holds, then

$$\lim_{n\to\infty}\frac{z_n}{n}=0, \qquad \lim_{n\to\infty}\frac{w_n}{n}=0;$$

and *a fortiori*

$$\lim_{n\to\infty}\frac{z_n}{2^n}=0, \qquad \lim_{n\to\infty}\frac{w_n}{2^n}=0.$$

It follows that (8) holds for each x in the interval $0<x<\pi/2$ except for a set having measure zero and that, under the proviso **B** above, Dobinski's formula (1) also holds.

5. Complex values of x. We finally consider the behavior of (1) when x is complex. Let $\theta=a+ib$. Then

$$\begin{aligned}
|\sin\theta| &= |\sin a\cosh b + i\cos a\sinh b| \\
&= [\sin^2 a\cosh^2 b+\cos^2 a\sinh^2 b]^{1/2} \\
&= [\sin^2 a\cosh^2 b+\cos^2 a(\cosh^2 b-1)]^{1/2} \\
&= [\cosh^2 b-\cos^2 a]^{1/2} \\
&= \cosh b\left[1-\frac{\cos^2 a}{\cosh^2 b}\right]^{1/2} \\
&= \tfrac{1}{2}e^b(1+e^{-2b})\left[1-\frac{\cos^2 a}{\cosh^2 b}\right]^{1/2}.
\end{aligned}$$

Putting $\theta=2^{n+1}x$, where $x=p+iq$, $q\neq 0$, we get

$$|\sin 2^{n+1}x|^{2^{-n}} = \left[\frac{1}{2}e^{2^{n+1}q}(1+e^{-2^{n+2}q})\left(1-\frac{\cos^2 2^{n+1}p}{\cosh^2 2^{n+1}q}\right)^{1/2}\right]^{2^{-n}}$$

$$= e^{2q}\left[\frac{1}{2}(1+e^{-2^{n+2}q})\left(1-\frac{\cos^2 2^{n+1}p}{\cosh^2 2^{n+1}q}\right)^{1/2}\right]^{2^{-n}}.$$

Hence

$$\lim_{n\to\infty}|\sin 2^{n+1}x|^{2^{-n}} = e^{2q}.$$

Combining this with an obvious modification of the argument following equation (9), we obtain the following general theorem.

Let x be any complex number, and θ any real number. Then values of $(\tan 2^n x)^{2^{-n}}$ for $n=1, 2, \cdots$ can be chosen so that

$$\lim_{n\to\infty}\tan x(\tan 2x)^{1/2}(\tan 4x)^{1/4}\cdots(\tan 2^n x)^{2^{-n}} = 4e^{2if(x)+i\theta}\sin^2 x,$$

provided that x does not belong to a certain set of measure zero on the real axis.

References

1. Dobinski, G. 1876. Product einer unendlichen Factorenreihe. Archiv der Mathematik und Physik, vol. 59, pp. 98–100. Dobinski's note in volume 59 of the Archiv der Mathematik und Physik bears the name G. Dobiciecki, but a correction following the index in the volume says the name should be G. Dobinski.
2. Borel, E. 1909. Les probabilities denombrables et leurs applications arithmetiques. Rendiconti del Circolo Matematico di Palermo, vol. 27, pp. 247–271.

ORTHOPOLAR AND ISOPOLAR LINES IN THE CYCLIC QUADRILATERAL

R. GOORMAGHTIGH, Bruges, Belgium

1. Introduction. According to a theorem given by Servais [1], the orthopoles of a straight line l with respect to the four triangles formed by three out of four vertices of any quadrilateral are on a straight line L, called the orthopolar line of l for the given quadrilateral. McBrien [2] has proved the same property and also mentioned several interesting envelopes of orthopolar lines in the case when the quadrilateral is cyclic.

It will be shown that these last results and some others may all be easily derived from two simple properties of the orthopolar line, which will also be extended to the isopolar line.

2. Two principal directions. Using complex coördinates, let t_1, t_2, t_3, t_4 be the turns corresponding to the vertices of a cyclic quadrilateral $A_1A_2A_3A_4$, inscribed in the base circle Γ having the center O, while Ω is the unit-point, and denote by

$\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the elementary symmetric functions of the t_i; that is,

$$\sigma_1 = \sum t_1, \qquad \sigma_2 = \sum t_1t_2, \qquad \sigma_3 = \sum t_1t_2t_3, \qquad \sigma_4 = \sum t_1t_2t_3t_4.$$

Parallels drawn through Ω to A_1A_2 and A_3A_4 cut Γ again at t_1t_2 and t_3t_4, so that the mid-points of the arcs on Γ between these two points are $\pm\sigma_4^{1/2}$.

This shows that the bisectors of the angles between two opposite sides, or between the diagonals of the quadrilateral are parallel to two perpendicular directions Δ, Δ', which is a well known property [3], and that, if $O\Omega$ is parallel to Δ, Δ', $\sigma_4 = 1$.*

3. Properties of the orthopolar lines. The projection of A_1 on the straight line l†

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1$$

being

$$\frac{a}{2} + \frac{t_1}{2} - \frac{a}{2\bar{a}t_1},$$

the orthopole of l with respect to $A_1A_2A_3$ is

$$x = \frac{1}{2}\left(a + t_1 + t_2 + t_3 + \frac{t_1t_2t_3\bar{a}}{a}\right)$$

or

$$x = \frac{1}{2}\left(a + \sigma_1 - t_4 + \frac{\bar{a}\sigma_4}{at_4}\right)$$

and, as

$$\bar{x} = \frac{1}{2}\left(\bar{a} + \frac{\sigma_3}{\sigma_4} - \frac{1}{t_4} + \frac{at_4}{\bar{a}\sigma_4}\right),$$

the elimination of t_4 leads to the equation of the orthopolar line L, namely,

$$2ax + 2\bar{a}\sigma_4\bar{x} = a\sigma_1 + \bar{a}\sigma_3 + a^2 + \sigma_4\bar{a}^2.$$

Suppose now that $O\Omega$ is parallel to Δ or Δ'; then the last equation becomes

$$2ax + 2\bar{a}\bar{x} = a(\sigma_1 + a) + \bar{a}(\bar{\sigma}_1 + \bar{a}).$$

Hence, we have two simple properties of the orthopolar line:

A given line l and its orthopolar line L for the cyclic quadrilateral have symmetric directions to those of the bisectors of the pairs of opposite sides and diagonals.

* These directions are those of the axes of the equilateral hyperbola circumscribed to the quadrilateral.

† $\bar{u}$ is the conjugate to u.

Further, L passes through the point $\sigma_1/2+a/2$; but $\sigma_1/2$ is the center ω of the equilateral hyperbola circumscribed to the quadrilateral, and $a/2$ is the projection of O on l.

The orthopolar line L contains the extremity of the segment equipollent to the distance from O to l, having as origin the center ω of the equilateral hyperbola circumscribed to the quadrilateral.

4. Envelopes of orthopolar lines. Suppose that l envelopes a curve of which the pedal curve as to O will be denoted by C. Let ωX and ωY be two straight lines forming angles with Δ and Δ' equal to $\pi/4$. If, for the center ω and the ratio 2, C' is homothetic to the curve derived from C by a translation of vector $O\omega$, then the envelope of L is also the envelope C_0 of the straight line joining the projections on ωX and ωY of a point moving on C'.

Various particular cases of this last envelope are well known. Some of these are listed below.

If a line l passes through a fixed point, C is a circle passing through O, and the orthopolar line L of l for the quadrilateral envelopes a deltoid.

When l passes through O, then L passes through ω.

It is also easy to prove that C_0 will reduce to a point when C' and C are equilateral hyperbolas passing respectively through ω and O and having their asymptotes parallel to ωX and ωY.

When l envelopes the antipedal curve as to O of an equilateral hyperbola passing through O and having its axes parallel to Δ and Δ', L passes through a fixed point.

Finally, according to a well known definition of the astroid, we find a generalization of a property given by McBrien [2]; namely,

When l envelopes a circle concentric to the circumcircle of the quadrilateral, L envelopes an astroid having ωX and ωY for cusp-tangents; the constant segment cut off by these on L is equal to the diameter of the circle, the envelope of l.

5. Extension to isopolar lines. The two properties of the orthopolar line, given in Section 3, may be extended to the isopolar line.

The straight lines,

$$\frac{x}{\bar{x}} = p_1 \qquad \frac{x}{\bar{x}} = \lambda p,$$

form an angle θ such that $\lambda = e^{2i\theta}$; hence a straight line through A_1, forming an angle θ with l, cuts l at the point

$$\frac{(-\lambda a + t_1 + a\lambda/\bar{a}t_1)}{(1-\lambda)}.$$

Further, the equation of the line through that point forming an angle $\pi-\theta\pi$ with A_2A_3 is

$$(1-\lambda)\left(x + \frac{t_2t_3\bar{x}}{\lambda}\right) = -a\lambda + t_1 + \frac{a\lambda}{\bar{a}t_1} + \frac{\bar{a}t_2t_3}{\lambda} - \frac{t_2t_3}{t_1} - \frac{\bar{a}t_1t_2t_3}{a\lambda}.$$

The isopole of l for the angle θ as to $A_1A_2A_3$ has, therefore, as coördinate

$$x = \frac{(-a\lambda + t_1 + t_2 + t_3 - \bar{a}t_1t_2t_3/a\lambda)}{(1-\lambda)}$$

or

$$x = \frac{(-a\lambda + \sigma_1 - t_4 - \bar{a}\sigma_4/a\lambda t_4)}{(1-\lambda)};$$

and, since

$$\bar{x} = \frac{(\bar{a} - \lambda\bar{\sigma}_1 + \lambda/t_4 + a\lambda^2 t_4/\bar{a}\sigma_4)}{(1-\lambda)},$$

the elimination of t_4 shows that the isopoles of l for the four triangles of the quadrilateral lie on the straight line

$$a\lambda^2 x + \bar{a}\sigma_4\bar{x} = \frac{(-a^2\lambda^3 + \bar{a}^2\sigma_4 + a\lambda^2\sigma_1 - \bar{a}\lambda\sigma_4\bar{\sigma}_1)}{(1-\lambda)};$$

when $O\Omega$ is parallel to Δ or Δ', the equation of the isopolar line becomes

$$\frac{ax + \bar{a}\bar{x}}{\lambda^2} = \frac{[a\sigma_1 - a^2\lambda + \bar{a}(\bar{a} - \lambda\bar{\sigma}_1)/\lambda^2]}{(1-\lambda)}.$$

The isopolar line for the angle θ of a line l forms an angle $2(\pi - \theta)$ with the symmetric of l as to the principal directions Δ and Δ' of the quadrilateral.

Further, as the isopolar line passes through the point

$$\frac{(\sigma_1 - a\lambda)}{(1-\lambda)},$$

we have the following property:

The isopolar line passes through the image of O in the point of the axis of the segment between ω and the projection of O on l, from which that segment subtends an angle 2θ.

References

1. Servais, Cl., Sur l'orthopôle, Mathesis, 1923. p. 11.

2. McBrien, V. O., On some systems of orthopolar and Kantor lines of lines referred to a quadrangle, The Catholic University of America Press, Washington, D. C., 1942.

3. -F. G.-M., Exercises de Géométrie, Tours, 7th ed., p. 288.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

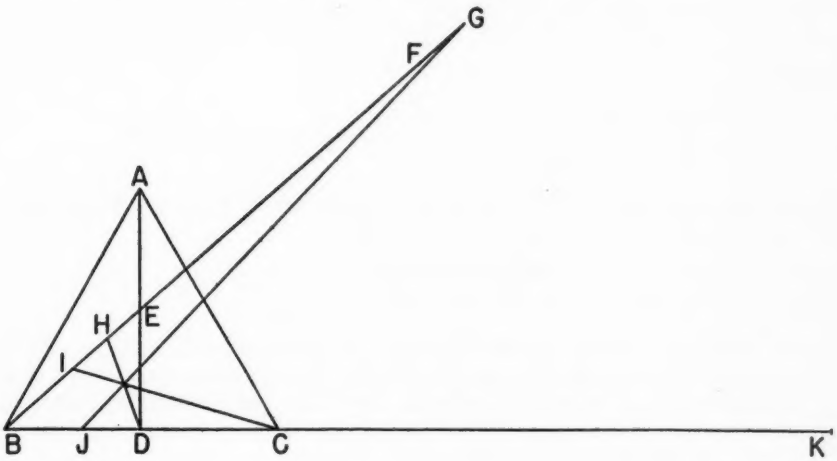
Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

AN APPROXIMATE CONSTRUCTION FOR e

H. F. SANDHAM, Trinity College, Dublin, Ireland

In the figure the triangle ABC is equilateral. The altitude AD is bisected at E. The line BE is produced to F and then to G so that $BF=2BC$, and $FG=\frac{1}{3}BE$. The points H and I are taken on BE so that $BH=BD$, and $BI=\frac{1}{3}BC$. The point J is taken on BC so that HD, IC, and JG are concurrent. The side BC is produced to K so that $CK=2BC$.

Then JK/BC is approximately equal to e.



COMPUTATION.
Since

$$AD = \frac{\sqrt{3}}{2}BC, \qquad DE = \frac{\sqrt{3}}{4}BC,$$

we have

$$BE = \frac{\sqrt{7}}{4}BC.$$

From

$$BF = 2BC, \qquad FG = \frac{\sqrt{7}}{12}BC,$$

it follows that

$$BG = \left(2 + \frac{\sqrt{7}}{12}\right) BC.$$

Also

$$GI = \left(\frac{5}{3} + \frac{\sqrt{7}}{12}\right) BC.$$

Now

$$(BJ/JC):(BD/DC) = (BG/GI):(BH/HI).$$

Hence

$$\frac{BJ}{JC} = \frac{24 + \sqrt{7}}{3(20 + \sqrt{7})}.$$

Therefore

$$\frac{JK}{BC} = \frac{228 + 11\sqrt{7}}{84 + 4\sqrt{7}} = \frac{4711 + 3\sqrt{7}}{1736} = 2.718,281,828,(30).$$

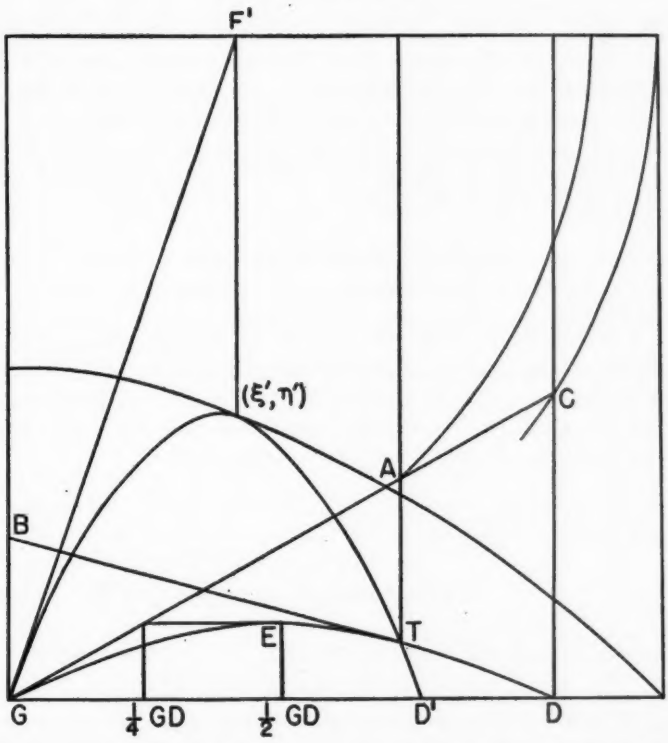
Thus since $e = 2.718,281,828,(46)$, the error is approximately 0.000,000,000,16.

THE TRAJECTORY IN VACUO

J. M. Thomas, Duke University

R. J. Walker, *An artillery problem*, this Monthly, vol. 54, 1947, pp. 33–34, has shown how to construct geometrically the angle of departure, the angle of fall, and the time of flight for a projectile fired in vacuo, the muzzle velocity v_0, the acceleration of gravity g and the positions of gun and target being known. His construction employs a fixed hyperbola. The present note gives for the above and related quantities ruler-and-compass constructions which the author used in lectures given during the late war. The discussion here is limited to a description of the constructions because elementary, simple proofs are easily made.

Let R be the maximum range. Use an accent to distinguish quantities belonging to the upper trajectory from those belonging to the lower. Denote by (a, b, r) the circle with center (a, b) and radius r. In a system of rectangular cartesian coördinates (see figure) plot the gun at $G = (0, 0)$ and the target at $T = (x_0, y_0)$. The distance GT is the slant range r. Let A, A' with ordinates a, a' be the intersections of $(x_0 - r, R, R - y_0)$ with the vertical $x = x_0$. Let B, B' be the points $(0, a - y_0)$, $(0, a' - y_0)$, which are most easily constructed by transferring the distances TA, TA'. Then GA, GA' and TB, TB' are the tangents to the trajectories at G and T, respectively. The times of flight are the lengths GA, GA', if the unit of length is made v_0 in reading them.



The intercepts of the trajectories on the x-axis are found by continuing GA, GA' until they intersect the circle $(0, R, R)$ in C, C' and then projecting C, C' onto the x-axis at D, D'.

Through the intersection of $x=\frac{1}{4}GD$ and GA draw a horizontal line. Its intersection with $x=\frac{1}{2}GD$ is the summit E of the trajectory.

Once D and E have been found, the familiar projective construction employed in mechanical drawing can be used to find any number of points on the trajectory as the intersections of corresponding rays of a pencil of equally spaced vertical lines and a pencil of rays drawn from the summit to equally spaced points on the vertical through G and that through D.

The envelope of the trajectories is the parabola which has vertex $(0, \frac{1}{2}R)$, has focus $(0, 0)$, and passes through $(R, 0)$. Points within or on the envelope are within range and can be hit. Points outside the envelope cannot be hit.

Let (ξ, η) be the point of contact of the lower trajectory with the envelope. Let GA meet $y=R$ in F. Let the line of departure complementary to GA meet the vertical through F at H. Then ξ is the abscissa of F and η is $\frac{1}{2}HF$.

The circle $(0, y_0-a, GA)$ is a curve of constant fuse setting. As the point (x_0, y_0) describes a trajectory, the center of this circle behaves like a particle

freely falling from rest at the gun. When the trajectories have been drawn, these circles can be used to parametrize them. Simultaneous positions of shells fired simultaneously on two trajectories can be compared by means of these circles.

There are other interesting loci connected with the problem. The envelope of the curves of equal fuse setting is the same as the envelope of the trajectories, the point of contact being imaginary if $v_0 t < R$. The locus of summits is the ellipse with vertices $(\pm\frac{1}{2}R, \frac{1}{4}R)$, $(0, 0)$, $(0, \frac{1}{2}R)$.

A NOTE ON INTERPOLATION

P. M. Hummel, University of Alabama

In a recent article* the following theorem was proved:

Let $f(x)$ be a function which together with its first three derivatives is continuous throughout the interval $a \leqq x \leqq b$. Moreover let $f''(x)$ and $f'''(x)$ be of constant sign throughout the interval. If a, b, $f(a)$, $f(b)$, and x are given and $f(x)$ is determined by linear interpolation, the error† satisfies the inequalities

$$\frac{(b-x)(x-a)}{(b-a)^2}\left[f(b) - f(a) - (b-a)f'(a)\right] \leqq \text{error} \leqq \frac{(b-x)(x-a)}{(b-a)^2}\left[(b-a)f'(b) - f(b) + f(a)\right],$$

or these inequalities reversed, according as $f'''(x)$ is positive or negative.

While the inequalities above give rather sharp limits for the error, they look somewhat formidable. The purpose of this note is to give alternate limitations for the error.

Using Taylor's finite expansion, one readily verifies that

$$f(b) - f(a) - (b-a)f'(a) = \tfrac{1}{2}(b-a)^2 f'(\theta_1), \qquad a < \theta_1 < b,$$
$$(b-a)f'(b) - f(b) + f(a) = \tfrac{1}{2}(b-a)^2 f'(\theta_2), \qquad a < \theta_2 < b.$$

When these results are substituted into the inequalities in the theorem, and obvious simplifications are made, we get

$$(1) \qquad \tfrac{1}{2}(b-x)(x-a)f''(\theta_1) \leqq \text{error} \leqq \tfrac{1}{2}(b-x)(x-a)f''(\theta_2), \quad a < \theta_1, \theta_2 < b.$$

Since $f''(x)$ is continuous, there exists a value θ, between θ_1 and θ_2, such that

$$(2) \qquad \text{error} = \tfrac{1}{2}(b-x)(x-a)f''(\theta), \qquad a < \theta < b.$$

Equation (2) is an exact expression for the error, and, by use of the extreme values of $f''(x)$ in the interval (a, b), gives limits for the error. This expression can be simplified further, though slightly weakened, by using the fact that the

* P. M. Hummel, The accuracy of linear interpolation, this Monthly, vol. 53, 1946, p. 364.

† The error due to interpolation is defined as *error = interpolated value − true value.*

quadratic factor $(b-x)(x-a)$ has a maximum value of $\frac{1}{4}(b-a)^2$. This fact enables us to write

$$|\,error\,| \leqq \frac{(b-a)^2}{8}\,|\,f''(\theta)\,|, \qquad a < \theta < b. \tag{3}$$

The relations (1), (2), and (3) are usually not as sharp as the inequalities given in the theorem, but in many cases are easier to use and are certainly easier to remember.

Finally it should be noted that all these results, including the inequalities in the theorem, do not include the error due to rounding off. Thus if $f(a)$ and $f(b)$ are not exact values but contain an error due to rounding off, then the interpolated value of $f(x)$ likewise contains an error due to rounding off. It is easy to show, however, that the rounding off error in $f(x)$ lies between the rounding off errors in $f(a)$ and $f(b)$.

NOTE

It has been pointed out by Paul D. Thomas that the method of evaluating $\int_0^\infty e^{-x^2}dx$ presented by H. F. Sandham, this MONTHLY, vol. 53, 1946, p. 587, had been found previously, and appears, for instance, in W. E. Byerly, *Integral Calculus*, second edition, 1888, p. 99.—E. F. B.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.

DERIVATION OF THE EQUATIONS OF CONICS

F. HAWTHORNE, Hofstra College

In most analytic geometry texts the equations of conics are derived by use of the distance formula. Alternative derivations which do not involve radicals seem to provoke some student interest. The procedure can be illustrated by the following derivation of the equation of the ellipse.

Let the foci be the points $(\pm c, 0)$; and let R be the distance from a point on the curve to the left-hand focus and r be the corresponding distance to the right hand focus. Then by definition:

$$R^2 = y^2 + (x+c)^2, \quad \text{and} \tag{1}$$

$$r^2 = y^2 + (x-c)^2. \tag{2}$$

Adding and subtracting we have:

$$R^2 + r^2 = 2(y^2 + x^2 + c^2), \quad \text{and} \tag{3}$$

$$R^2 - r^2 = 4cx = (R + r)(R - r). \tag{4}$$

But by definition of the ellipse

$$R + r = 2a. \tag{5}$$

Hence from (4) and (5) we have that

$$R - r = 2cx/a. \tag{6}$$

Adding and subtracting (5) and (6) gives:

$$R = a + cx/a, \quad \text{and} \tag{7}$$

$$r = a - cx/a. \tag{8}$$

Putting the results of (7) and (8) into (3) gives:

$$a^2 + \frac{c^2x^2}{a^2} = c^2 + x^2 + y^2, \tag{9}$$

from which the standard equation is found by putting $b^2 = a^2 - c^2$. The expressions for the focal radii in terms of the eccentricity may be pointed out in equations (7) and (8).

A SUBSTITUTION FOR SOLVING TRIGONOMETRIC EQUATIONS

R. W. WAGNER, Oberlin College

The substitution $t = \tan(\theta/2)$ which is used in the calculus to integrate some expressions containing both $\sin\theta$ and $\cos\theta$ can also be used to advantage in the solution of equations in which the unknown appears only as the argument of one or more trigonometric functions. The usefulness of this substitution arises from the fact that the trigonometric functions of θ are simple rational functions of t. From

$$\tan(\theta/2) = t,$$

one gets, by using the double angle formula,

$$\tan\theta = \frac{2t}{(1 - t^2)},$$

and hence that

$$\sin\theta = \frac{2t}{\sqrt{(2t)^2 + (1 - t^2)^2}} = \frac{2t}{1 + t^2}.$$

There is no ambiguous sign in this last equation. For $\sin\theta$ and t have the same algebraic sign. Also, one gets

$$\cos\theta = \frac{1-t^2}{1+t^2}.$$

The neatness of this method of solution will be shown by applying it to an example:

EXAMPLE. *Find the angles between* $-\pi$ *and* π *radians such that*

$$5\sin\theta - 21\cos\theta = 10.$$

On making the above substitutions one gets

$$\frac{10t}{1+t^2} - 21\frac{1-t^2}{1+t^2} = 10.$$

When this equation is cleared of fractions and terms collected the result is

$$11t^2 + 10t - 31 = 0.$$

Hence,

$$t = \frac{-10 \pm \sqrt{100+1364}}{22}$$

$$= 1.285 \quad \text{or} \quad -2.194.$$

From a table of trigonometric functions in radian measure one finds that

$$\theta/2 = 0.910 \quad \text{or} \quad -1.143,$$

and that

$$\theta = 1.820 \quad \text{or} \quad -2.286 \text{ radians.}$$

The main advantage of this method of solving the given equation lies in the fact that there is no difficulty in determining the quadrant in which an angle may lie. It is easy to bear in mind that negative values of t correspond to negative angles and positive values of t correspond to positive angles.

EVALUATION OF A TRIGONOMETRIC INTEGRAL

J. P. HOYT, United States Naval Academy

The purpose of this note is to present a quick and easy method for evaluating $\int \sin^{2m} x \cos^{2n} x\,dx$, where m and n are positive integers. No table of formulas is required.

To illustrate the method consider the integration of $\int \sin^4 x \cos^6 x\,dx$. Since $\sin nx = (1/2i)(e^{inx} - e^{-inx})$ and $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$, we let $y = e^{ix}$, and find that $\sin nx = (1/2i)(y^n - 1/y^n)$ and $\cos nx = \frac{1}{2}(y^n + 1/y^n)$. Then

$$\sin^4 x = \frac{\left(y - \frac{1}{y}\right)^4}{2^4} = \frac{y^4 - 4y^2 + 6 - \frac{4}{y^2} + \frac{1}{y^4}}{2^4}.$$

If we notice that the powers of y decrease by 2 from term to term, we may write this expression in the following symbolic form:

$$\sin^4 x = \frac{1-4+6-4+1}{2^4}.$$

Now instead of expanding $\cos^6 x$ in the same manner and then forming the product, it is easier and quicker to multiply the expansion of $\sin^4 x$ by the expression for $\cos x$ six times in succession. For the coefficients of each expansion are obtained from the coefficients of the preceding one in the same manner as the coefficients of $(a+b)^{n+1}$ are obtained from the coefficients of $(a+b)^n$; that is, in the form used in Pascal's Triangle.

The successive expansions appear as follows in the symbolic notation:

$$\begin{aligned}
\sin^4 x &= (1-4+6-4+1)/2^4 \\
\sin^4 x \cos x &= (1-3+2+2-3+1)/2^5 \\
\sin^4 x \cos^2 x &= (1-2-1+4-1-2+1)/2^6 \\
\sin^4 x \cos^3 x &= (1-1-3+3+3-3-1+1)/2^7 \\
\sin^4 x \cos^4 x &= (1+0-4+0+6+0-4+0+1)/2^8 \\
\sin^4 x \cos^5 x &= (1+1-4-4+6+6-4-4+1+1)/2^9 \\
\sin^4 x \cos^6 x &= (1+2-3-8+2+12+2-8-3+2+1)/2^{10}
\end{aligned}$$

or

$$\sin^4 x \cos^6 x = \left(y^{10}+2y^8-3y^6-8y^4+2y^2+12+\frac{2}{y^2}-\frac{8}{y^4}-\frac{3}{y^6}+\frac{2}{y^8}+\frac{1}{y^{10}}\right)\Big/2^{10}.$$

Regrouping the last expression we obtain:

$$\sin^4 x \cos^6 x = \left[\left(y^{10}+\frac{1}{y^{10}}\right)+2\left(y^8+\frac{1}{y^8}\right)-3\left(y^6+\frac{1}{y^6}\right)-8\left(y^4+\frac{1}{y^4}\right)+2\left(y^2+\frac{1}{y^2}\right)+12\right]\Big/2^{10}$$

which can be rewritten:

$$\sin^4 x \cos^6 x = (\cos 10x + 2\cos 8x - 3\cos 6x - 8\cos 4x + 2\cos 2x + 6)/2^9.$$

So immediately we have the final result:

$$\int \sin^4 x \cos^6 x\,dx = \left[\frac{\sin 10x}{10}+\frac{\sin 8x}{4}-\frac{\sin 6x}{2}-2\sin 4x+\sin 2x+6x\right]\Big/2^9+C.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 759 [1947, 107]. *Corrected. Proposed by Theodore Running, Ann Arbor, Michigan*

Show that $x^n-(x-a)^n$ can be expressed as the difference of two squares in at least one way for all positive integral values of n, x, and a, a less than x, not counting the obvious way when n is even.

E 766. *Proposed by H. E. G. P.*

Professor Umbugio, who was introduced to our readers in our foregoing April number, invented a remarkable scheme for reviewing books. He divides the time he allows himself for reviewing into three fractions, α, β, and γ. He devotes the fraction α of his time to a deep study of the title page and the jacket. He devotes the fraction β to a spirited search for his name and for quotations from his works. Finally, he spends the fraction γ of his alloted time in a proportionally penetrating perusal of the remaining text. Knowing his characteristic taste for simple and direct methods, we cannot fail to be duly impressed by the differential equations on which he bases his scheme:

$$(1) \qquad dx/dt = y - z, \qquad dy/dt = z - x, \qquad dz/dt = x - y.$$

He considers a system of solutions x, y, z which is determined by initial conditions depending on a (small) parameter ϵ, independent of t. Therefore x, y, and z depend on both t and ϵ, and we appropriately use the notation:

$$(2) \qquad x = f(t, \epsilon), \qquad y = g(t, \epsilon), \qquad z = h(t, \epsilon).$$

The functions (2) satisfy the equations (1) and the initial conditions

$$(3) \qquad f(0, \epsilon) = 1/3 - \epsilon, \qquad g(0, \epsilon) = 1/3, \qquad h(0, \epsilon) = 1/3 + \epsilon.$$

Professor Umbugio defines his important fractions α, β, and γ by

$$\lim_{\epsilon \to 0} f(2, \epsilon) = \alpha, \qquad \lim_{\epsilon \to 0} g(5, \epsilon) = \beta, \qquad \lim_{\epsilon \to 0} h(279, \epsilon) = \gamma.$$

Deflate the Professor! Find α, β, γ without much numerical computation.

E 767. *Proposed by Milton Schwartz, Temple University*

From any point P on BC of parallelogram $ABCD$ line segments are drawn to A and D. From any point Q on AD line segments are drawn to B and C. Through the intersections of these four segments (PA, PD, QB, QC) a line is drawn meeting AB in R and CD in S. Prove that BR equals DS.

E 768. *Proposed by Irving Kaplansky, University of Chicago*

A number n has the property that for any $p<q<n$,

$$S = p + (p+1) + \cdots + q$$

is never divisible by n. Show that this is true if and only if n is a power of 2.

E 769. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a plane quadrangle $ABCD$, the perpendicular at A to side AB cuts the opposite side CD in M, and the perpendicular at A to side AD cuts the opposite side BC in N. Show that the radical axis of the circles described on AM and AN as diameters coincides with the tangent at A to the equilateral hyperbola circumscribing the quadrangle.

E 770. *Proposed by Norman Miller, Queen's University, Canada*

A function $f(x)$ has the following definition, n being a positive integer,

$$f(x) = x^n \sin \pi/x, \qquad x \neq 0,$$
$$f(0) = 0.$$

Show that, if $n=2m$ or $2m+1$ (m a positive integer), then at $x=0$ the function possesses a derivative of the mth but no higher order, and that, at $x=0$, $f^{(m)}(x)$ is continuous or discontinuous according as $n=2m+1$ or $n=2m$.

SOLUTIONS

A Characteristic Root

E 733 [1946, 394]. *Proposed by E. D. Schell, Arlington, Va.*

Show that a square matrix of order n, whose elements form a magic square, has a characteristic root equal to $n(n^2+1)/2$.

Solution by the Proposer. Denote the vector consisting of a row of n ones by u. Since the sums s of the n columns of the matrix A are equal, $uA=us$. This may be written as $u(A-sI)=0$, where I is the identity matrix. Hence s is a characteristic root of A.

The sum of the first n^2 integers is $n^2(n^2+1)/2$, which is the sum of all n columns of the matrix. Hence the sum of a single column is $s=n(n^2+1)/2$.

Also solved by D. W. Alling, Alfred Brauer, and Daniel Finkel.

Alfred Brauer points out that this problem is a special case of the following theorem proved in his paper, *Limits for the characteristic roots of a matrix* (Duke Mathematical Journal, vol. 13 (1946), pp. 387–395).

Let $A=(a_{\kappa\lambda})$ be a square matrix of order n, $R_\kappa=\sum_{\lambda=1}^{n}|a_{\kappa\lambda}|$, and $R=\max(R_1, R_2, \cdots, R_n)$. The absolute value of one of the characteristic roots of A equals R only if

$$R_1 = R_2 = \cdots = R_n = R \tag{1}$$

and

(2) $$\arg(a_{\alpha\lambda}) = \phi + \phi_\alpha - \phi_\lambda,$$

where ϕ and $\phi_1, \phi_2, \cdots, \phi_n$ are arbitrary real numbers. If (1) and (2) hold, then $Re^{i\phi}$ is a characteristic root.

Non-Linear Spring Problem

E 734 [1946, 394]. *Proposed by Henry Scheffé, University of California at Los Angeles*

A body in the form of a plate is supported by a smooth horizontal surface, which we take as the x, y-plane. A vertical peg is fastened to the body through its center of gravity. Two similar springs of natural length l run from this peg to pegs fixed in the plane, one at $(0, l)$, and the other at $(0, -l)$, the springs being parallel to the plane. The body is released from rest in a position where its center of gravity is on the x-axis. Assuming the forces exerted by the springs proportional to their elongations, and the masses of the springs negligible, show that for small oscillations the frequency is proportional to the amplitude.

I. *Solution by P. T. Bateman, Yale University.* Let k be the constant of each spring, m the mass of the body, and v the velocity of the body. If the body is at the point $(x, 0)$, the force of each spring is

$$k\{(x^2 + l^2)^{1/2} - l\}.$$

The components in the y-direction cancel, but the components in the x-direction add up to

$$2k\{(x^2 + l^2)^{1/2} - l\}\frac{-x}{(x^2 + l^2)^{1/2}} = \frac{-2kx^3}{\{(x^2 + l^2)^{1/2} + l\}(x^2 + l^2)^{1/2}}.$$

Thus for small x we have approximately

$$mv\frac{dv}{dx} = \frac{-2kx^3}{2l^2} = \frac{k}{l^2}x^3.$$

Integration gives $v^2 = k(a^4 - x^4)/(2ml^2)$, where a is the amplitude. Thus the period corresponding to the amplitude a is

$$T_a = 4\int_0^a \frac{dx}{\{k(a^4 - x^4)/(2ml^2)\}^{1/2}} = 4l\left(\frac{2m}{k}\right)^{1/2}\int_0^a \frac{dx}{(a^4 - x^4)^{1/2}}.$$

Hence

$$T_{ta} = 4l\left(\frac{2m}{k}\right)^{1/2}\int_0^{ta} \frac{dx}{(t^4a^4 - x^4)^{1/2}} = 4l\left(\frac{2m}{k}\right)^{1/2}\int_0^a \frac{tdw}{(t^4a^4 - t^4w^4)^{1/2}} = \frac{T_a}{t}.$$

II. *Solution by Elmer Latshaw, ACF-Brill Motors Co., Philadelphia, Pa.* Let θ be the angle between an extended spring and its initial or free position, and $R/2$ the scale of a spring. Then the resultant of the tensions in the stretched springs is along the x-axis and is equal to

$$Rl(\tan\theta - \sin\theta) = Rl\left(\theta + \frac{\theta^3}{3} + \cdots - \theta + \frac{\theta^3}{6} - \cdots\right),$$

or, for small oscillations, approximately equal to $Rx^3/2l^2$. The differential equation of motion of the weight W, for small oscillations, may then be taken as

$$\left(\frac{W}{g}\right)\ddot{x} + \left(\frac{R}{2l^2}\right)x^3 = 0,$$

or

$$\ddot{x} + \frac{\phi x^3}{2} = 0, \quad \text{where} \quad \phi = \frac{gR}{Wl^2}.$$

The solution of this differential equation is

$$x = \frac{v^{1/2}\,\mathrm{sn}\,(v^{1/2}t\phi^{1/4})}{\phi^{1/4}\,\mathrm{dn}\,(v^{1/2}t\phi^{1/4})}, \tag{1}$$

where $t=0$ when $x=0$, v is the velocity at $x=0$, and the modulus of the elliptic functions is $k=\sqrt{2}/2$. (See Michael A. Sadowsky's article in the December 1945 issue of the *Journal of the Franklin Institute* for a derivation of the general solution of our differential equation, and of numerous other equations arising from non-linear spring problems. It is, of course, an easy matter to check the solution by substituting it in the differential equation.) Differentiating (1) we find

$$\dot{x} = v\frac{\mathrm{cn}\,(v^{1/2}t\phi^{1/4})}{\mathrm{dn}^2\,(v^{1/2}t\phi^{1/4})}.$$

At maximum amplitude A, occurring at time T, we require that $\dot{x}=0$. Therefore, at maximum amplitude A we must have

$$\mathrm{cn}\,(v^{1/2}T\phi^{1/4}) = 0,$$

$$\mathrm{sn}\,(v^{1/2}T\phi^{1/4}) = 1,$$

$$\mathrm{dn}\,(v^{1/2}T\phi^{1/4}) = \sqrt{2}/2.$$

From the first of these it follows that

$$v^{1/2}T\phi^{1/4} = \tfrac{1}{2}K = 1.854, \text{ a constant.} \tag{2}$$

From (1) we have

$$A = \frac{\sqrt{2}\,v^{1/2}}{\phi^{1/4}},$$

and from (2)

$$v^{1/2} = \frac{c}{T\phi^{1/4}}.$$

Eliminating v we find

$$A = \frac{\sqrt{2}\,c}{T\phi^{1/2}} = \frac{4\sqrt{2}\,cf}{\phi^{1/2}},$$

where f is the frequency. Thus, for small oscillations, the frequency is proportional to the amplitude.

Also solved by W. J. Nemerever and T. H. Simester.

This problem furnishes a simple example on small vibrations in which the frequency is *not* independent of the amplitude.

Isosceles n-Points

E 735 [1946, 394]. *Proposed by Paul Erdös, Stanford University*

Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in space which cannot be so arranged?

Solution by L. M. Kelly, University of Missouri. We shall refer to a set of n points, each of whose triples forms an isosceles triangle, as an *isosceles n-point.* We propose first to establish that some point of a plane isosceles 6-point must have the property that at least three of the segments joining it to the remaining five points are equal. That is, that at least three of the segments radiating from some vertex are equal.

Certainly some vertex, say 1, has at least two equal radiating segments. Let $12=13=a$, $14=x$, $15=y$, $16=z$, and let us suppose that no vertex has the desired property. Then $x\neq a$, $y\neq a$, $z\neq a$. Furthermore, from triangle 124, we conclude that either $24=a$ or $24=x$. Similarly, from triangle 134, $34=a$ or $34=x$. Both 34 and 24 cannot equal x, since then vertex 4 would satisfy the desired condition. Thus either 24 or 34 must equal a. No loss of generality is entailed if we assume $34=a$. Then, in triangle 135, 35 must equal y. Again, triangle 125 implies that $25=a$, and triangle 136 implies that $36=z$. Finally, triangle 126 presents us with a dilemma, since 26 must equal a or z, and in either case we have a contradiction of the assumption. Thus at least one vertex must have the announced property. That is, an isosceles 6-point must possess an isosceles triangle and its circumcenter.

We now wish to prove that a plane isosceles 6-point cannot contain certain special configurations.

1. *Three linear points.* Suppose 1, 2, 3 linear, with $12=23=a$. Then 24 must equal a, otherwise angles 124 and 324 would both be base angles of isosceles triangles, which is impossible. This implies that angle 143 is a right angle and triangle 134 is right isosceles. Any other point 5, which together with 1, 2, 3, 4 forms an isosceles 5-point, must be the reflection of 4 in 13. We thus see that any plane isosceles 5-point containing a linear triple forms the vertices and center of a square. Clearly, then, no plane isosceles 6-point with a linear triple is possible.

2. *Equilateral triangle and circumcenter.* Suppose 4 the circumcenter of equi-

lateral triangle 123. Set $12=13=23=a$, $14=24=34=b$. Consider a fifth point 5 forming, if possible, with 1, 2, 3, 4 an isosceles 5-point. Now $15\neq 25$, since otherwise 3, 4, 5 would be linear. From triangle 125, either $15=a$ or $25=a$. We may assume $25=a$. Similar considerations show that either $35=a$ or $15=a$, and in either case there results a linear triple.

3. *Two equilateral triangles with common base.* Let $12=23=13=43=14=a$, $24=b$. Suppose, as before, a fifth point 5. Now $15=35$ would imply a linear triple. So, from triangle 135, we conclude either $15=a$ or $35=a$. Assume, without loss, $35=a$. Also $45\neq 25$ and, hence, either $45=b$ or $25=b$. We may again assume $45=b$. Triangle 145 then demands that $15=b$. Now triangle 125 provides a contradiction, since 25 must equal a or b, which is impossible.

4. *Configuration with* $12=13=23=34=a$, $14=24=b$, $b<a$. Consider a fifth point 5. Now $15\neq 25$, since otherwise 3, 4, 5 would be linear. Therefore, from triangle 125, either $15=a$ or $25=a$. Assume, without loss, that $25=a$. Now, from triangle 245, either $45=a$ or $45=b$. If $45=a$, 5 would be the reflection of 3 in 42, and a simple computation shows that triangle 145 is not isosceles. Thus $45=b$. Triangle 345 compels $35=b$, since if $35=a$ configuration 3 would result. Finally, $15=b$. Now the 5-point is completely labeled and can easily be seen to be impossible of realization in the plane.

5. *Configuration with* $12=13=23=34=a$, $14=24=b$, $b>a$. Consider a fifth point 5. Now $15\neq 25$, since otherwise configuration 1 would result, and $15=a$ or $25=a$. Suppose, without loss, that $25=a$. Observe that $35\neq a$, since otherwise configuration 3 would result. Thus, since $15\neq a$, we have, from triangle 135, that $35=15$. From this it follows that $45\neq a$, since otherwise angle 135 would be obtuse. From triangle 245, 45 must equal b. Hence $35=15=b$. This causes 1, 3, 4, 5 to form configuration 4.

Characterization of plane isosceles 6-point. From our opening result we know that a plane isosceles 6-point must contain the circumcenter of one of the isosceles triangles. Let $12=13=14=a$, $24=34=b$, $23=c$. We know that $a\neq b$, since otherwise configuration 3 would result; $a\neq c$, since otherwise configuration 5 would result; $c\neq b$, since otherwise configuration 2 would result. Also, $25\neq 35$, since otherwise configuration 1 would result. Therefore, from triangle 235, we see that either $25=c$ or $35=c$. Suppose, without loss, that $35=c$. We now maintain that 15 must equal a. For suppose not. Then, from triangle 135, $15=c$, and from triangle 125, $25=a$. We now try to label 45. Triangle 245 implies that $45=b$ or a; triangle 345 implies that $45=b$ or c; triangle 145 implies that $45=a$ or c. These conditions are mutually impossible. Thus $15=a$. Now $45\neq b$, since otherwise 1, 4, 3 would be linear. Hence, from triangle 345, $45=c$. Finally, $25=b$. Examination of this configuration reveals that 2, 3, 4, 5 are four of the five vertices of a regular pentagon with 1 as its center. We can now conclude that the only possible sixth point must be the remaining vertex, and we have characterized plane isosceles 6-points as the vertices and center of a regular pentagon.

A necessary consequence of the above characterization of plane isosceles 6-points is that there can be no plane isosceles 7-points.

In three space the above kind of analysis would become much more difficult.

However, we can show that the answer to the question for three space cannot be less than nine. For consider a regular pentagon and its circumcenter. Take a point above the plane of the pentagon directly over the center and at a distance equal to the radius of the pentagon from it. Also consider the reflection of this point in the plane. These eight points form a space isosceles 8-point.

Editorial Note. Associated with the problem of characterizing the maximum isosceles n-points in euclidean spaces is the characterization of the maximum two distance sets in euclidean spaces, that is, the characterization of the maximum number of points in a euclidean space such that the various distances consist of no more than two distinct numbers. Kelly has shown that the maximum two distance set in the plane consists of five points which form the vertices of a regular pentagon. This set, along with its circumcenter, constitutes a plane isosceles 6-point, and, in the above solution, Kelly shows this set to be the maximum plane isosceles n-point. From this one might be led to conjecture that the maximum isosceles n-point in euclidean r-dimensional space would be the vertices, constituting a two distance set, of an r-dimensional polytope inscribed in a sphere, along with the center of the sphere. H. S. M. Coxeter asserts that the polytopes of this kind in 3, 4, 5 dimensions which he believes to be the best possible are those called "pure Archimedean" in his first published paper: *Proc. Camb. Phil. Soc.*, 24 (1928), pp. 1–9. (Coxeter remarks that these were described earlier by Thorold Gosset, *Messenger of Math.*, 29 (1900), pp. 43–48.) The one in 3 dimensions is the equilateral triangular prism with square sidefaces. The one in 4 dimensions has 10 vertices, the midpoints of the 10 edges of the regular simplex. The one in 5 dimensions has 16 vertices, which are alternate vertices of the 5-dimensional hyper-cube. A polytope of this kind in 6 dimensions is the subject of a more recent paper (with some fine pictures) by Coxeter: *The polytope 2_{21} whose 27 vertices correspond to the lines on the general cubic surface*, Amer. J. of Math., 62 (1940), pp. 457–486.

Of course, the concluding paragraph of Kelly's solution shows that the above conjecture is not true. Nevertheless, Coxeter's examples for the higher spaces probably constitute the best results yet known for these dimensions.

A Property of k Positive Integers

E 736 [1946, 462]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2<\cdots<a_k\leqq n$, where $k>[(n+1)/2]$, be k positive integers. Then $a_i+a_j=a_r$ is solvable. (*Cf.* No. 3739 [1937, 120].)

Solution by Leo Moser, University of Toronto. The $k-1$ positive integers $a_2-a_1, a_3-a_1, \cdots, a_k-a_1$, are clearly all distinct. These, together with the k given distinct a's, give $2k-1>n$ positive integers, each not greater than n. Hence at least one integer is common to both sets, so that at least once $a_r-a_1=a_i$, or $a_i+a_1=a_r$.

Also solved by P. T. Bateman, P. A. Clement, William Gustin, J. B. Kelly, and the proposer. The proposer pointed out that the sequence

$$[n/2]+1,\ [n/2]+2,\cdots,n$$

shows that for $k=[(n+1)/2]$ the result is false.

Divergence of a Series

E 737 [1946, 462]. *Proposed by V. L. Klee, Jr., University of Virginia*

Establish the divergence of the series $\sum_{n=1}^{\infty} n^{-a}\cos(b\log n)$ for all values of $a\leqq 1$, regardless of the value of b.

Solution by Norman Miller, Queen's University, Canada. If $b=0$, the series clearly diverges. Take $b\neq 0$. Then $\lim_{n\to\infty}|b|\log n=\infty$ and $\lim_{n\to\infty}[b\log(n+1)-b\log n]=0$. Hence, given any positive integer M, we can find a positive integer p so large that between $(p-1/3)\pi$ and $(p+1/3)\pi$ there are at least M consecutive values of $|b|\log n$. For all of these values $\cos(b\log n)$ has the same sign and $|\cos(b\log n)|>1/2$. Hence the sum of the corresponding M consecutive terms of the given series is in absolute value greater than $\frac{1}{2}\sum_{n=q}^{q+m} n^{-a}$ for some integer q. Since this sum can be made arbitrarily large by a proper choice of M, the series fails to satisfy the Cauchy criterion for convergence.

Also solved by P. T. Bateman, Max Wyman, and the proposer. Wyman utilized the Cauchy integral test; Bateman and the proposer showed, somewhat as above, that the Cauchy criterion for convergence is not satisfied. Bateman pointed out that the divergence of the given series implies the well known divergence of $\sum_{n=1}^{\infty} n^{-z}$, $R(z)\leqq 1$. This is easily seen since, for real b, $\sum n^{-a}\cos(b\log n)=\sum R\{n^{-(a+ib)}\}$.

A Property of the Monge Point

E 738 [1946, 462]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the four spheres passing through the Monge point and the nine point circles of the faces of a tetrahedron are equal to each other.

Solution by the Proposer. Let O, G, Ω be the circumcenter, centroid, and Monge point of the tetrahedron $ABCD$, and let M, N, P, Q, S, T be the midpoints of the edges BC, CD, DB, AB, AC, AD.

The symmetrics A', B', C', D', with respect to G, of the vertices A, B, C, D are vertices of tetrahedra $A'MNP$, $B'NST$, $\cdots$, inversely homothetic to $ABCD$ in the ratio $-1/2$. The tetrahedron $A'MNP$ is symmetric, with respect to G, to the tetrahedron $ASTQ$, whence the circumsphere of the first passes through Ω, the symmetric, with respect to G, of the circumcenter O, and the points A' and Ω are diametrically opposite on this sphere. The theorem now follows.

Otherwise, let A'' be the point diametrically opposite to A on the circumsphere. The $A'\Omega$ is the transform of AA'' under the homothety $(G_a, -1/2)$, where G_a is the centroid of the face BCD. Again the theorem follows.

Editorial Note. The four equal spheres under consideration have radii equal to half that of the circumsphere, and are anologous to the circles $B'OC'$, $C'OA'$,

$A'OB'$ of a triangle ABC, where O is the circumcenter and A', B', C' are the midpoints of the sides BC, CA, AB.

Inscribed Ellipse

E 739 [1946, 462]. *Proposed by L. M. Kelly, University of Missouri*

An ellipse inscribed in the triangle ABC is tangent to AB at D. Show that the midpoints of CD and AB are collinear with the center of the ellipse.

I. *Solution by H. E. Fettis, Dayton, Ohio.* Let P and Q be the midpoints of AB and CD, and let O be the center of the ellipse. Draw the diameter DO of the ellipse and let the tangent at the other end, G, of this diameter cut AC in M and BC in N.

AN and BM intersect in a point S which lies on DG and also on CP. Then, if CG intersects AB in H, we have, from similar triangles,

$$AD/GN = AS/SN = AB/MN,$$

$$BH/GN = BC/CN = AB/MN.$$

Therefore $AD = BH$, or P is the midpoint of DH, so that OP is parallel to GH and bisects CD at Q.

Otherwise. The result may be considered as a special case of the well known fact that the centers of all conics inscribed in a given quadrilateral are collinear with the midpoints of the diagonals of the quadrilateral. In the special case when two sides of the quadrilateral coincide, we have that the centers of all conics inscribed in a given triangle, and touching one side of the triangle at a fixed point, are collinear with the midpoint of that side and the midpoint of the line joining the opposite vertex with the fixed point of contact. This last statement is equivalent to the proposition stated in the problem.

II. *Solution by Joseph Rosenbaum, Milford School, Connecticut.* Project the ellipse orthogonally into a circle. We are then confronted with the corresponding theorem for a circle inscribed in a triangle. Preserving the lettering, consider the triangle $A'B'C$, where A', B' are the intersections with CA and CB of a line parallel to AB and tangent to the incircle (O) of ABC at E'. Since (O) is the excircle of triangle $A'B'C$ for the side $A'B'$, the intersection E of CE' with AB is the point of contact of the excircle of triangle ABC for side AB. Hence the midpoint M of AB is also the midpoint of DE. This, together with the fact that O is the midpoint of DE', implies the conclusion of the problem.

Also solved by E. F. Allen, J. C. Currie, J. M. Kingston, J. H. Simester, the proposer, and jointly by J. H. Butchart, Ressa Olson, Jean Rowe, and Jacquelin Wilbanks. The joint solution and the solutions by Currie and the proposer employed orthogonal projection as in solution II. The remaining solutions were analytic, Allen establishing the result for any central conic touching the three sides of a triangle.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4244. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider squares of $2n$ digits which can be formed by bringing together two consecutive n-digit numbers. Show that in every system of numeration there is at least one unending sequence of such squares. Show also that, if n and B are given and if x^2 is such a square, then y^2 is another where $x+y=B^n+1$ (provided squares with initial zeros are admitted). As a particular example let $B=12$, $n=3$.

4245. *Proposed by J. H. Butchart, Arizona State College*

The envelopes of two families of lines, PQ, PQ', making angles of $\pm 30°$ respectively with the tangents to a deltoid at their points of contact P are two deltoids, larger than the given one in the ratio $3^{1/2}:1$. Show also that $PQ=PQ'$, where Q, Q' are the points of contact of PQ, PQ' with the respective envelopes, and that the angles between the cusp tangents of the envelopes and the included cusp tangent of the given deltoid are $\pm 10°$.

4246. *Proposed by J. A. Greenwood*

Evidently $y'=x'Dy$, $y''=(x''D+x'^2D^2)y$, $\cdots$, where $D\equiv d/dx$ and primes indicate differentiation with respect to t. Find an explicit expression for $y^{(n)}$.

4247. *Proposed by Leon Recht and Martin Rosenman, New York City*

The sequence

$$\{a\} = 1; 2; 1, 2; 2, 1, 2; 1, 2, 2, 1, 2; \cdots$$

is formed by writing down in succession sets of terms starting with 1 and such that every subsequent set is obtained from the preceding set by the substitution $1\rightarrow 2; 2\rightarrow 1, 2$. Show that: (1) the nth term a_n is

$$[k(n+1)] - [kn],$$

where $k=\frac{1}{2}(\sqrt{5}+1)$ and $[x]$ is the greatest integer not exceeding x; (2) Each set of terms in $\{a\}$, beginning with the third, is a repetition of the terms of the two preceding sets; (3) $\{a\}$ consists of sets of 2's separated by 1's, such that there are a_n 2's in the nth set of 2's.

4248. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $ABCD$ be a given tetrahedron. Place a sphere (S) of given radius in such

a manner that the volume of the polar tetrahedron of $ABCD$ with respect to (S), will be a minimum.

SOLUTIONS

The Tangential Triangle

4179 [1945, 523]. *Proposed by J. R. Musselman, Western Reserve University*

The poles of the medians of a triangle $A_1A_2A_3$ as to its circumcircle are three points of a line; those points where the external bisectors of the angles of the tangential triangle of $A_1A_2A_3$ meet the opposite sides of this tangential triangle.

Solution by H. E. Fettis, Dayton, Ohio. The problem is not quite accurately stated unless $A_1A_2A_3$ is an acute-angled triangle; for the statement to hold in general, the words "external bisectors" should be replaced by "angle bisectors (of the tangential triangle) not passing through the circumcenter of $A_1A_2A_3$."

Let the vertices of the tangential triangle be designated by B_1, B_2, B_3. Then B_1M_1, where M_1 is the midpoint of A_2A_3, passes through the circumcenter of $A_1A_2A_3$, and B_1 is the pole of A_2A_3. If P_1 is the pole of A_1M_1, then B_1M_1 and B_1P_1 are polar conjugate lines relative to B_1A_2 and B_1A_3, and are therefore harmonically separated from B_1A_2 and B_1A_3.

Therefore, since B_1M_1 is a bisector of angle $A_2B_1A_3$, B_1P_1 must be the conjugate bisector. That is, the bisectors of angles of the tangential triangle not passing through the circumcenter of the given triangle pass respectively through the poles of the medians of the given triangle. These poles are obviously on the sides of the tangential triangle and since the medians are concurrent, their poles are necessarilly collinear.

Solved also by J. W. Clawson, Howard Eves, R. Goormaghtigh, A. S. Howard, Ou Li, and A. Sisk.

Editorial Note. Goormaghtigh calls attention to the following generalization: If $A_1A_2 \cdots A_n$ is a polygon inscribed in a circle Γ having O as center, the poles of the joins of any of the vertices A_i to the centroids G_i of the remaining vertices are on a straight line; those points are the intersections of the tangents at the points A_i to the circle Γ with the perpendiculars erected at the inverse points T_i of the points G_i as to Γ on the lines OT_i.

Goormaghtigh also refers to the remarkable equation given by J. H. Weaver (this MONTHLY, 1933, 91) for the straight line g containing the points of intersection of the sides of the triangle $B_1B_2B_3$ with the external bisectors of that triangle. If, in a system of complex coördinates, $A_1A_2A_3$ is the base-circle and if σ_1 is the sum of the coördinates of A_1, A_2, A_3, the equation of g is

$$x\bar{\sigma}_1 + \bar{x}\sigma_1 = 6,$$

$\bar{a}$ being conjugate to a. But the polar of a point α as to the base-circle $x\bar{x}=1$ is

$$x\bar{\alpha} + \bar{x}\alpha = 2;$$

hence g is the polar of the centroid $\sigma_1/3$. Corresponding to his generalization for

the n-gon we have the following generalization of Weaver's equation: If $A_1A_2 \cdots A_n$ is a polygon inscribed in a circle Γ having O as center, and if G_i is the centroid of the vertices other than A_i and T_i the inverse of G_i as to Γ, the tangents to Γ at the points A_i meet the perpendiculars erected at the points T_i to the lines OT_i on the straight line

$$x\bar{\sigma}_1 + \bar{x}\sigma_1 = 2n,$$

σ_1 being the sum of the coördinates of the vertices A_i in a system of complex coordinates in which Γ is the base-circle.

The proposer suggests the following extension: The poles of the symmedians of a triangle are the centers of the Appolonian circles and lie on the Lemoine axis.

Envelope of Conics

4182 [1945, 582]. *Proposed by Cezar Coşniţa, Focşani, Roumania*

Show that the envelope of the conics circumscribing a given triangle and such that the angle between the asymptotes is constant, is a curve of the fourth degree bitangent to the line at infinity at the circular points and having the vertices of the triangle for double points.

I. *Solution by G. A. Williams, Oregon State College.* Let the equations of the asymptotes be $y-m_1x+\alpha=0$ and $y-m_2x+\beta=0$ with the condition

$$\frac{m_2 - m_1}{1 + m_1m_2} = \text{constant} = \frac{1}{k}, \tag{1}$$

and let the vertices of the given triangle be $(0, 0)$, $(a, 0)$, (b, c). Then the equation of the conic may be written

$$(y - m_1x + \alpha)(y - m_2x + \beta) = \lambda. \tag{2}$$

Upon substituting the coördinates of the given points in (2) and then eliminating λ, α, β, and m_1, a quadratic in m_2 is obtained:

$$Am_2^2 + Bm_2 + C = 0, \tag{3}$$

where

$$\begin{aligned} A &= kcx^2 - cxy - kacx + b(ka - kb + c)y, \\ B &= -cx^2 - 2kcxy + cy^2 + acx + (b^2 - ab - c^2 + 2kbc)y, \\ C &= cxy + kcy^2 - c(kc + b)y. \end{aligned}$$

The eliminant of this quadratic and its partial derivative with respect to m_2 equated to zero is $\phi \equiv B^2-4AC=0$. Then as each of A, B, C vanishes for $(0, 0)$, $(a, 0)$, (b, c), it follows that $\phi=0$, $\partial\phi/\partial x=0$, $\partial\phi/\partial y=0$ are all satisfied at each point and hence the curve has a double point at each.

Upon expanding ϕ we see that $\phi=c^2(x^2+y^2)^2$ plus terms of lower degree. Hence the line at infinity meets the curve twice at each of the circular points.

As the curve can have no more double points than the three already obtained, the line at infinity must be tangent at each of the circular points.

II. *Solution by L. M. Kelly, University of Missouri.* The proposed theorem is merely the result of applying the transformation by isogonal conjugates with respect to the given triangle to the following: The envelope of equal chords of the circumcircle is a circle concentric with the circumcircle.

Under such a transformation the circumcircle goes over into the line at infinity. A family of equal chords goes into a family of similar and circumscribed conics. Since the transformation is a contact transformation the circle which is the envelope of the chords will go into the envelope of the conics. But a circle concentric with the circumcircle goes into a curve of the fourth degree. Since the circular points are isogonal conjugates with respect to any triangle and since two concentric circles are tangent at the circular points, we see that the desired locus is a fourth degree curve tangent to the line at infinity at the circular points. Finally, since the envelope of the chords cuts each side of the triangle in two points, the fourth degree curve must have double points at each vertex.

Also solved by H. E. Fettis, R. Goormaghtigh, and G. B. Huff.

Sums of Powers with Binomial Coefficients

4183 [1945, 582]. *Proposed by Cezar Coşniţa, Focşani, Roumania*

Let p and q be non-negative integers and x a variable. Define

$$f(x, p, q) = \sum_{i=0}^{p} (-1)^i {}_pC_i (x - i)^q,$$

where ${}_pC_i$ are binomial coefficients. Prove that $f(x, p, q)$ equals zero if $p>q$; equals $p!$ if $p=q$; and is a polynomial in x of degree $q-p$ if $p<q$.

I. *Solution by F. Underwood, University College, Nottingham, England.* Put

$$\begin{aligned} A = (e^{at} - e^{bt})^p &= e^{pat} - {}_pC_1 e^{[(p-1)a+b]t} + {}_pC_2 e^{[(p-2)a+2b]t} - \cdots \\ &= e^{xt} - {}_pC_1 e^{(x-1)t} + {}_pC_2 e^{(x-2)t} - \cdots, \end{aligned}$$

in which we have substituted $pa=x$, $a-b=1$, (whence $pb=x-p$). The coefficient of t^q in the expansion of A in powers of t is

$$\frac{1}{q!}\left[x^q - {}_pC_1(x-1)^q + {}_pC_2(x-2)^q - \cdots\right] = \frac{f(x, p, q)}{q!}.$$

On the other hand A may be put in the form

$$\begin{aligned} A &= \left[e^{(a-b)t} - 1\right]^p e^{pbt} = (e^t - 1)^p e^{(x-p)t} \\ &= \left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]^p e^{(x-p)t} \\ &= t^p \left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right]^p \left[1 + (x-p)t + \frac{(x-p)^2 t^2}{2!} + \cdots\right]. \end{aligned}$$

In this form of A the coefficient of t^q is zero when $p>q$, unity when $p=q$, a polynomial of degree $q-p$ in x when $p<q$. The equality of the corresponding coefficients completes the proof.

II. *Solution by E. A. Hedberg, University of South Carolina.* Let $a_0=0$, $a_1, a_2, \cdots, a_p$ be any set of distinct numbers, real or complex, and put

$$ {}_pG_i = (-1)^{i+p}\prod_{j=1}^{p}\frac{a_j}{a_i-a_j}, \qquad j\neq i,\ i=1, 2, \cdots, p. $$

If we define

$$ F(x, p, q) = \sum_{i=0}^{p}(-1)^i\,{}_pG_i(x-a_i)^q, $$

then $F(x, p, q)$ is zero when $p>q$, equals $\prod_{i=1}^{i=p}a_i$ when $p=q$, and is a polynomial of degree $q-p$ in x when $p<q$. (The ${}_pG_i$ enjoy many of the properties of the ${}_pC_i$ and indeed reduce to them if we take $a_i=i$. Thus the ${}_pG_i$ may be considered as a natural generalization of binomial coefficients.) The proof follows.

$F(x, p, q)=\sum_{i=0}^{i=p}(-1)^i\,{}_pG_i(x-a_i)^q$ is identically equal to D_1/D_2, where D_1 is the determinant

$$ \begin{vmatrix} x^q & (x-a_1)^q & (x-a_2)^q & (x-a_3)^q & \cdots & (x-a_p)^q \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & a_1 & a_2 & a_3 & \cdots & a_p \\ 0 & a_1^2 & a_2^2 & a_3^2 & \cdots & a_p^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_1^{p-1} & a_2^{p-1} & a_3^{p-1} & \cdots & a_p^{p-1} \end{vmatrix} $$

and D_2 is the determinant of Vandermonde which results when the first row and first column of D_1 are deleted.

In case $p>q$, all elements of the first row of D_1 become zero if we subtract from it the products of the second row by x^q, the products of the third row by $-{}_qC_1x^{q-1}$, of the fourth row by ${}_qC_2x^{q-2}$, and so on. Hence $F(x, p, q)=0$.

In case $p=q$, if the foregoing procedure is applied, and the proper interchange of rows is made to bring the first row into the position of the last, D_1 becomes

$$ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_1 & a_2 & \cdots & a_p \\ 0 & a_1^2 & a_2^2 & \cdots & a_p^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_1^{p-1} & a_2^{p-1} & \cdots & a_p^{p-1} \\ 0 & a_1^p & a_2^p & \cdots & a_p^p \end{vmatrix} = \prod_{i=1}^{p}a_iD_2. $$

That is, $F(x, p, p) = \prod_{i=1}^{q} a_i$.

When $p < q$, it is seen that the $(q-p)$th derivative of $F(x, p, q)$ reduces to $q(q-1) \cdots (p+1)F(x, p, p)$ which is a constant not equal to zero. Hence $F(x, p, q)$ is of degree $(q-p)$ in x.

Also solved by David Alling, W. J. Combellack, Sidney Glusman, H. A. Luther, and the Proposer.

Squares of Special Form

4184 [1945, 582]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what systems of numbers with the base less than 10000 are there the greatest number of squares of four digits of the form $aabb = (cc)^2$? Dedicated to E. P. Starke.

Solution by Fritz Herzog, Michigan State College. Let m denote the base $(m \geq 2)$. Assume a, b, c to be non-negative integers less than m such that

$$am^3 + am^2 + bm + b = (cm + c)^2. \tag{1}$$

We disregard the trivial solution $0000 = 00^2$ and assume $a+b$ and c to be positive. (1) may be put in the form

$$a(m - 1) + (a + b)/(m + 1) = c^2. \tag{2}$$

Since $0 < a+b < 2m+2$ we conclude from (2) that

$$a + b = m + 1. \tag{3}$$

From (2) and (3) we obtain

$$a(m - 1) + 1 = c^2, \tag{4}$$

$$c^2 \equiv 1 \pmod{m - 1}. \tag{5}$$

From (3) we conclude that $a \neq 0, 1$, and hence by (4)

$$c \neq 1, \quad \sqrt{m}. \tag{6}$$

It is easily seen, on the other hand, that any integer c $(1 \leq c \leq m-1)$ which satisfies (5) and (6), together with a and b determined by (4) and (3), yields a solution of the problem. Therefore, if we denote by $A(m)$ the number of solutions of the problem for the base m, we have

$$A(m) = B(m - 1) - \delta, \tag{7}$$

where $B(n)$ denotes the number of solutions of the congruence $x^2 \equiv 1 \pmod{n}$, and δ equals 2 or 1 according as m is or is not a perfect square (see (6)). If s denotes the number of distinct odd prime factors of n, the value of $B(n)$ is given by

$$B(n) = 2^{s+2}, \ 2^{s+1}, \text{ or } 2^s, \tag{8}$$

according as $n \equiv 0 \pmod 8$, $\equiv 4 \pmod 8$, or $\not\equiv 0 \pmod 4$, respectively.* With (7)

* See, for instance, Landau, *Vorlesungen über Zahlentheorie*, vol. 1, p. 47, Satz 88.

and (8) the number of solutions of the problem for each base is completely determined.

For $n<9999$ we have $s\leqq 4$ (since $3\cdot 5\cdot 7\cdot 11\cdot 13>10000$) and hence the greatest value of $B(n)$ is $2^{4+2}=64$ corresponding to $n=8\cdot 3\cdot 5\cdot 7\cdot 11=9240$. For $m=n+1=9241$, we have $\delta=1$ and $A(m)=63$. Evidently all other $m<10000$ give $A(m)\leqq 31$.

Also solved by A. Barriol, Free Jamison, and E. P. Starke.

Editorial Note. The original proposal set the upper limit of 100000 for m. However Barriol showed that there is no new value of m for which $A(m)$ exceeds 63. For the following numbers $A(m)=63$: 60061, 78541, 87781, 92421.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics of Finance. By J. A. Northcott. New York, Rinehart & Co., Inc., 1946. 9+252 pages. $3.00.

In the author's own estimate, this text is a "brief and simple development of a subject of great importance to students of banking and finance." He feels that a defense is necessary for his presumption in adding another text to the field. The defense offered is that most of the standard texts place undue emphasis on a multiplicity of formulas—a policy that the present text seeks to supplant by the systematic use of a small body of fundamentals.

The reviewer frequently overhears students discussing mathematics of finance. Their attitude is characterized by one admonition: "Don't take that course! It's all story problems and they have formulas this long!" To students, then, and to their instructors as well, an apology for a text which achieves, to any appreciable degree, brevity, simplicity and system, must seem pointless.

The first two chapters comprise a résumé of progressions and logarithms. The treatment covers—with plenty to spare—all possible applications of these two topics to the subject at hand.

The treatment of simple interest, which many writers draw out to needless length, covers less than six pages, and only one formula is introduced. It is noteworthy that the so-called simple discount rate d, that is, the interest per year per dollar of amount, is omitted. From the standpoint of minimum confusion to the student, this trifling detail d, whose sole excuse for existence is its aid to computation, might be profitably ignored. One usually hesitates, however, to deal so summarily with a concept as widely used as this one. Some authors

include, as exercises in simple interest, transactions involving cash payments made at different dates. The treatment of these problems by the methods of simple interest is both cumbersome and unnatural; the only satisfactory way of handling them is through equation of value with compound interest. Our author is to be commended for adhering to the latter policy.

The author defines compound interest as simple interest in which the principal is converted at the end of each conversion period. Historically, this is probably the way that compound interest came into being, and it is certainly the popular conception of the topic. It is far neater, however, to define the compound interest law as a value-*vs*-time relation $V(t_2) = V(t_1)k^{t_2-t_1}$, where $V(t)$ is the value at date t and k is constant. Then we need give no thought to "principal" and "amount"; to whether we are going forward or backward in time; to whether the time interval is a whole number of conversion periods. Furthermore, when we speak of two equivalent rates (j, m) and (j', m'), we are actually referring to the same compound interest law, rather than to two laws which agree only at certain periodic dates. The popularly conceived form of compound interest is then easily introduced as a computational detail. The reviewer has found that this method of presentation, aided by graphical study, gives the student a better insight than the other approach.

The powerful method of equation of value is given moderate emphasis—but still not enough. The author tacitly assumes that this method of solving a problem (in which all items of investment are brought, at compound interest, to an arbitrary reference date, summed, and equated to a like sum for the returns) will give the same result as analysis by amortization schedule (a progressive, item-by-item study). These methods are conceptually distinct (the amortization schedule being prior to the other approach) and their equivalence should be demonstrated. It is surprising how many students sense this point and demand just such an explanation.

The author foregoes the double superscript notation for annuity symbols. As a result, the user of his text is thrown back almost entirely upon the geometric progression formula. This "roughing it" is undoubtedly healthy for the student—if he doesn't lose heart and quit. Why not show the student once and for all how the progression formula gives rise to the more refined, mechanical procedure of the double superscript notation, and then let him put his progressions back on the reserve shelf?

The author introduces no separate symbolism or formulas for the treatment of annuities due and deferred annuities. Unfortunately, however, he allots a section to each. This invariably deludes the student into thinking that here are two new topics which he must learn (with consequent injury to his morale). How is he to know that there is just as much point in classifying annuities as "due," "immediate" and "deferred" as in distinguishing between the numbers 16, 17-1, and 24-8?

The treatment of bonds is typical of the average text. In order to correlate this topic with the mathematical tools previously developed, it is important to

prevent the reader from imagining that a bond is essentially different from any other set of investments and returns. The present text may fall slightly short in this respect.

Sinking funds and amortization are presented together in the traditional way. This presentation effectively conceals an important characteristic of situations involving sinking funds—namely, that they involve the interrelation of several distinct investors, each with his own investor's rate and his own set of payments.

A short chapter on depreciation and a chapter of miscellaneous review problems complete the text portion. There are 105 pages of tables, including 5 place logarithms and tables for compound interest and annuity computations.

The exercises are generally well fitted to the text, and they are stated with better than average clarity. Some of them are taken from actuarial examinations. Answers are furnished for the odd-numbered exercises.

An instructional device that our author has not fully utilized is graphical representation. The line diagram, which he uses freely, is admirably adapted to the itemizing of investments and returns; but the concept of *variation of invested value*, which underlies the whole subject of investment mathematics, can be best imparted to the student through a series of two-dimensional graphs. For example, the student should be invited to draw a few graphs of items at simple interest, to show the linearity of the value law; of items at compound interest, to distinguish the true and the approximate value; of bond price, to show the periodic variation and the general trend.

On the whole, the author has succeeded moderately well in achieving his set purpose of systematizing finance calculations. However, there are still many important principles that need enunciation. For example, what is the general method of treating commercial paper in analyzing a transaction? Perhaps the only way to change finance mathematics from an art to a science is to put it on a mildly postulational basis, and to effect a small degree of formalization. Most authors, including the present one, make desultory attempts along this line, but invariably their precepts remain tacit.

G. F. Rose

Mathematician's Delight. By W. W. Sawyer. New York, Penguin Books, Inc., 1946. 215 pages. $0.25.

This book is the eighth in the new series of Pelican Books, a series of small inexpensive books planned to treat a wide variey of subjects. The book under review provides, in many repects, an excellent general introduction to elementary mathematics. The title of the book is indicative of the author's ingenuity and originality in departing from traditional and time-worn patterns. In his own words, the book is intended to "try to show what mathematics is about, how mathematicians think, when mathematics can be of some use." Although intended chiefly for beginners, there is much in the book that will be of interest to the teacher as well.

In discussing the fear of mathematics which is held by so many laymen, the author contends that this is due not to the nature of the subject itself, but to the dull way in which it is so often taught. The thesis is developed that in many cases the student does not learn mathematics, but only an imitation of mathematics, which destroys his power to enjoy the real subject. In this connection Dr. Sawyer pays his respects to parrot-learning by quoting the pupil who wrote, "The abdomen contains the stomach and the vowels, which are A, E, I, O, and U!"

The book contains an interesting chapter on geometry in which the value of experimentation is emphasized and a number of rather ingenious geometrical experiments are proposed. A chapter, *The Nature of Reasoning*, leads gradually to mathematical reasoning and to the nature of mathematical abstraction. In making generalizations the author may occasionally be guilty of over-simplification. For example, the statement that "The more one studies the methods of the great, the more common-place do these methods appear," may require considerable qualification. An interesting discussion contrasting pure and applied mathematics is given and the point is made that all mathematics is tied, however remotely, to problems which arose originally in the study of the real world. This will seem rather academic to those mathematicians who believe that many of these ties are very remote indeed.

A chapter entitled *The Strategy and Tactics of Study* contains many interesting and sensible remarks. Subsequent chapters deal with arithmetic, logarithms, algebraic operations, functions, graphs, calculus, and trigonometry. All of these topics are introduced in an intuitive and discursive way, and in many cases quite unusual illustrative examples are used. The reviewer believes that in a few instances the ingenuity with which the author has rigged up his introductory illustrative examples may tend to make obscure to the beginner the mathematical principles involved. This is believed to be particularly true in connection with the way in which logarithms are introduced, where the mathematical principle involved seems much simpler than the mechanical device used by the author to provide a setting for the principle.

The chapter on trigonometry goes as far as the differentiation of the sine and cosine functions and applies the results to the study of the motion of a particle in a circle. It seems unfortunate that the important application of trigonometry to wave motion is not discussed. Following the chapter on trigonometry are two concluding chapters, one entitled *On Backgrounds*, and the other *The Square Root of Minus One*. The former chapter emphasizes the importance of understanding what lies behind the steps of a formal proof and in knowing what it is all about. In the last chapter imaginary numbers are treated from an operational point of view.

While in various places throughout the book the reader may not agree wholly with the author's contentions, nevertheless the reviewer regards this little book as a worthy addition to the growing list of survey type books, which deal in an elementary and intuitive way with the nature and the scope of mathe-

matics while leaving its rigorous elaboration and methodology to the text-books and treatises.

H. P. EVANS

NEW BOOKS RECEIVED

Advanced Mathematics for Engineers. Second Edition. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1947. 12+508 pages. $5.00.

An Introduction to Mathematical Genetics. By Lancelot Hogben. New York, W. W. Norton and Co., Inc., 1946. 12+260 pages. $5.00.

Mathematics of Finance. By F. S. Harper. Scranton, Pa., International Textbook Company, 1946. 10+327 pages.

Science Since 1500. By H. T. Pledge. New York, Philosophical Library, 1947. 357 pages. $5.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE.—In line with the policy of this department to suggest methods or ways of creating interest in mathematics among students and to print suggestions or ideas which might be used as projects for clubs, the following contribution by Mr. Frank Hawthorne is presented. Mr. Hawthorne has been a frequent contributor to the Problems and Solutions sections of this MONTHLY.

The editor of this department welcomes other suggestions which may be passed on to members of Mathematics Clubs.

INCREASING STUDENT INTEREST IN MATHEMATICS

One of the objectives stated or implied in the constitutions of most collegiate mathematics clubs is the fostering of increased interest in mathematics among the general student body. On most campuses there are many students who do not take mathematics courses or belong to mathematics clubs. Of course, some of these students contend that they "don't like mathematics" and will resist any attempt to expose them to it but there are usually others who have some interest and ability along that line. An attempt to increase the interest of these particular students may result in a few pleasant surprises.

As a possible means of reaching these people, selected problems whose solution requires some thought but only very elementary technique may be posted upon the general bulletin board at regular intervals. Specific times and places for discussion of each set of problems should be arranged. The number and diversity of people who attend these meetings may indicate that mathematics does have a certain general appeal. Perhaps some of the problems might even interest a few faculty members from non-mathematical departments. At the very least

one may hope that a few students will realize that such problems compare favorably with crossword puzzles in providing mental exercise.

Suitable problems are available in numerous books on recreational mathematics and additional problems of a sufficiently elementary nature appear from time to time in mathematical journals. Consideration might be given to the following problems which have appeared recently in this MONTHLY:

E576 (1944, 95), E581 (1944, 166), E608 (1944, 531), E622 (1945, 96), E631 (1945, 219), E651 (1945, 397), E671 (1946, 41), E751 (1947, 38).

E. J. Moulton's *A Problem in Geography* (1944, 216 and 220) intrigues many ex-navigators. The solution need not be complicated by the necessity for an argument about the non-existance of bears of any color in the Antarctic.

In selecting problems, care should be taken to avoid using too many of the same type. There is little danger that problems will be "old stuff" to those for whom they are particularly intended. Certainly those people will not have seen them in the MONTHLY.

Some problems simpler than those noted above should be used while an occasional relatively difficult problem might be included. Some examples (by no means new) follow:

1. An electric toaster will toast two slices of bread on one side each simultaneously. If three or any greater odd number of slices is desired, what procedure will use the least electricity?

2. What digits can be final digits of perfect squares? Cubes?

3. A cube three inches on an edge is made from white wood and all faces are painted black. It is then sawed into one inch cubes.

   a. How many saw cuts were necessary?
   b. How many small cubes will there be?
   c. How many small cubes will be all white?
   d. How many small cubes will have only one, only two, three black faces?

4. A board has three holes, one circular, the second square, the third equilaterally triangular. The diameter of the circle, the side of the square, and the altitude of the triangle are equal. Design a solid which will pass through each and fill each completely.

Occasionally a mildly facetious problem might be proposed. Such "problems" as "If one man can see five miles, how far can three men see?" or "If one man can do a job in an hour, how long will it take five hundred men to do it?" might season the selection. Certain students might even answer the second of these on the basis of recent experience. Of course, trick problems whose solutions hinge on a double meaning of words or some other equally obnoxious device should be avoided.

A general invitation to students to submit problems for posting and discussion at a later date should produce some results. Many of the problems so proposed may be unsuitable but some of them may prove helpful. It might be well to recall that many significant additions to mathematics have been made by persons whose interest was first aroused by some particular problem.

NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

THE INSTITUTE OF MATHEMATICAL STATISTICS

This society announces the election of the following officers: President, William Feller; Vice-presidents, J. H. Curtiss and M. H. Hanson; Secretary, Paul Dwyer.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1947:

Brown University (Graduate Division of Applied Mathematics). From July 7 to August 1 the following courses will be offered: by Professor Carrier, theory of structural stability; by Professor Lin, dynamics of viscous fluids. From August 5 to August 30: by Professor Goldstein, dynamics of compressible fluids; by Professor Prager, theory of plasticity.

The Catholic University of America. From June 27 to August 9 the following graduate courses will be offered: by Professor E. J. Finan, higher algebra; by Dr. V. McBrien, theory of equations; by Professor O. J. Ramler, college geometry, differential equations and analytic projective geometry; by Professor J. N. Rice, statistics and advanced calculus.

Columbia University. From July 7 to August 15 the following graduate courses will be offered: by Professor Eilenberg, homotopy theory; by Professor Kasner, survey of modern mathematics, geometry and dynamics; by Professor Lorch, theory of groups, Fourier series; by Professor Murray, differential equations; by Professor Smith, theory of functions of a real variable.

Northwestern University. From June 23 to August 23 the following advanced courses will be offered: definite integrals, functions of a complex variable, geometry for teachers, history and teaching of mathematics, introduction to the theory of groups, introduction to the theory of numbers, seminar in analysis, theory of equations, theory of statistics, theory of tensors.

Ohio State University. From June 17 to August 29 the following advanced courses will be offered: by Professor Alden, introduction to the theory of functions of a complex variable, differential equations; by Professor Helsel, advanced calculus, advanced geometry; by Professor Mann, fundamental ideas in algebra and geometry, theory of fields.

Stanford University. From June 19 to August 30 the following advanced courses will be offered: by Professor Rademacher (of the University of Pennsylvania), elementary mathematics from a higher point of view, elliptic functions; by Professor Szegö, partial differential equations of physics and engineering, another course to be announced.

Teachers College, Columbia University. From July 7 to August 15 the following courses will be offered: by Professor Clark, teaching algebra in secondary

schools, teaching arithmetic in the elementary schools; by Professor Fehr, professionalized subject matter in junior high school mathematics, professionalized subject matter in advanced secondary school mathematics; by Dr. Lazar, history of mathematics, logic for teachers of mathematics; by Mr. Mirick, elementary mechanics (statics), observation and participation in the teaching of geometry; by Professor Reeve, teaching and supervision of mathematics—junior high school, teaching and supervision of mathematics—senior high school; by Professor Schlauch, business mathematics; by Professor Shuster, teaching geometry in secondary schools, field work in mathematics. In addition, on consecutive Thursdays beginning on July 10, there will be given five special lectures and discussions in which all the instructors above and others will take part.

The University of Colorado. In both terms (June 16 to July 17 and July 21 to August 22 respectively) the following advanced courses will be offered: by Professor Kempner, teachers' course in mathematics (not a methods course); by Professor Hutchinson, functions of a real variable; by instructors not yet determined, theory of equations, vector analysis.

The University of Michigan. From June 23 to August 15 the following advanced courses will be offered in addition to the standard courses in differential equations, theory of equations, advanced calculus, mechanics and statistics: by Professor Bartels, vector analysis and hydrodynamics; by Professor Brauer, elementary matrices, theory of group representations; by Professor Coburn, operational mathematics; by Professor Copeland, foundations of mathematics, mathematical probability; by Professor Craig, significance tests and analytic sampling theory; by Professor Dushnik, theory of integration; by Professor Dwyer, computational methods; by Professor Hay, advanced mechanics; by Professor Kaplan, elementary functions of a complex variable with applications; by Professor Karpinski, teaching of geometry, history of algebra; by Professor Myers, functions of a real variable; by Professor Nesbitt, mortality studies; by Professor Rainich, introduction to differential geometry, higher geometry; by Professor Rainville, intermediate differential equations; by Professor Rothe, partial differential equations; by Professor Samelson, general spaces; by Professor Thrall, algebraic theory.

The University of Minnesota. From June 16 to July 25 the following advanced courses will be offered: by Professor Cameron, Fourier, Bessel and Legendre series, course in reading and research; by Professor Hatfield, differential equations, theory of numbers; by Professor Wegner, advanced calculus, solid analytic geometry. From July 28 to August 29: by Professor Olmsted, vector analysis; by Professor Wegner, advanced calculus, theory of matrices.

The University of North Carolina. From June 12 to July 22 the following advanced courses will be offered: by Professor Browne, introduction to the theory of matrices; by Professor Cameron, introduction to higher algebra; by Professor Garner, history of mathematics; by Professor Henderson, finite groups; by Professor Hill, elementary mathematical statistics; by Professor Hobbs, theory of equations; by Professor Linker, differential equations; by Professor Mackie, ad-

vanced calculus. From July 23 to August 29: by Professor Brauer, theory of equations (continued); by Professor Hoyle, differential equations (continued); by Professor Lasley, synthetic projective geometry; by Professor Winsor, college geometry; by Professor Wong, advanced calculus (continued).

The University of Pennsylvania. From June 30 to August 23 the following graduate courses will be offered: by Professor Beal, analytic geometry of three dimensions; by Professor Caris, diophantine analysis; by Professor Clarkson, theory and practice of approximation; by an instructor not yet decided, higher calculus.

The University of Virginia. From June 30 to August 23 the following advanced courses will be offered: by Professor Hedlund, advanced calculus and applied mathematics, foundations of geometry; by Professor Harrold, transformation topology.

The University of Wyoming. From June 9 to August 16 the following semi-graduate and graduate courses will be offered: by Professor Barr, methods of teaching mathematics; by Dr. Bristow, projective geometry; by Dr. Schwid, ordinary differential equations; by Dr. Varineau, advanced college algebra. From June 23 to August 25: by Miss Neubauer, the history of mathematics; by Dr. Smith, college geometry.

EXAMINATION ANNOUNCED FOR STATISTICIAN POSITIONS

An examination has been announced by the Civil Service Commission for filling high-grade professional Statistician positions in Washington, D. C., and vicinity. The salaries range from $5,905 to $9,975 a year.

To qualify, candidates must have had progressively responsible professional experience in statistical research. Graduate study with major work in statistics will be credited as being equivalent to professional work. No written test is required; applicants will be rated on their experience and training relevant to the duties of the positions. The grade or salary level for which applicants are considered qualified will be determined by the quality of their experience as shown by the scope and level of the responsibilities involved, their influence on policy and program, and the complexity of the problems handled. The age limit of 62 years is waived for persons entitled to veteran preference.

Applications will be accepted in the Commission's Washington office until further notice. However, persons interested in being considered for positions which are to be filled immediately should apply within one month. Application forms may be obtained from first- and second-class post offices, from the Commission's regional offices, or direct from the U. S. Civil Service Commission, Washington 25, D. C.

PERSONAL ITEMS

President L. R. Ford and his wife represented the Association at the Rockford College Conference on February 21–23, 1947.

Professor P. K. Rees of Louisiana State University represented the Association at the installation of Robert Cecil Cook as president of Mississippi Southern College.

R. H. Beard, of the New York Telephone Company has received the 1947 award of the Duodecimal Society of America.

W. D. Lambert of the U. S. Coast and Geodetic Survey has been elected Corréspondent of the Paris Academy of Sciences (Institut de France) in the Section of Geography and Navigation.

Associate Professors Garret Birkhoff and Saunders MacLane of Harvard University have been promoted to professorships.

Associate Professor J. J. Barron of Marquette University has been appointed to a professorship at Marshall College, Huntington, West Virginia.

Dr. O. K. Bower of the University of Illinois has been promoted to an assistant professorship.

A. H. Bowker has been appointed to an assistant professorship at Stanford University.

Professor Gregory Breit of the University of Wisconsin has been appointed to a professorship at Yale University.

Dr. J. W. Calkin of the California Institute of Technology has been appointed to an associate professorship at Rice Institute.

Associate Professors E. A. Cameron and V. A. Hoyle of the University of North Carolina have been promoted to professorships.

Dr. Harold Chatland of the University of Montana has been appointed to an assistant professorship at Ohio State University.

Paul Cramer of Huron College, South Dakota, has been appointed to an assistant professorship at Monmouth College, Illinois.

Professor Tobias Dantzig of the University of Maryland has retired.

Professor J. L. Doob of the University of Illinois has been appointed to a visiting professorship at Columbia University.

Dr. R. H. Downing of Fleetwings, Inc., Bristol, Pennsylvania, has been appointed to a professorship at the Army Air Forces Institute of Technology, Wright Field.

Associate Professor C. M. Erikson of Michigan State Normal College has been promoted to a professorship.

Dr. B. E. Gatewood has been appointed to an associate professorship at the Army Air Forces Institute of Technology, Wright Field, Dayton, Ohio.

J. B. Greeley has been appointed chairman of the mathematics department at Utica College, Syracuse University.

Assistant Professor J. F. Heyda of Franklin and Marshall College has been appointed research mathematician at the Naval Ordnance Plant, Indianapolis.

Professor Theodore Lindquist of Michigan State Normal College has retired.

Dr. A. N. Milgram has been appointed to an associate professorship at Syracuse University.

G. J. O'Boyle of the Catholic University of America has been promoted to an assistant professorship.

Associate Professor E. K. Paxton of Washington and Lee University has resigned.

Dr. Paul Reichelderfer has been appointed to an associate professorship at Ohio State University.

Dr. R. W. Shephard of the University of California has been appointed to an assistant professorship at New York University.

Dr. F. C. Smith of the Lincoln National Life Insurance Company has been appointed to an associate professorship at the College of St. Thomas, St. Paul, Minnesota.

Henry E. Smith of Dickinson College has been promoted to an assistant professorship.

Professor H. E. Spencer of Presbyterian College, Clinton, South Carolina, has been appointed to an assistant professorship at Virginia Polytechnic Institute.

Dr. George Tunell, on leave from the Carnegie Institution of Washington, has been appointed acting associate professor of mineralogy and metalliferous geology at California Institute of Technology.

Professor H. S. Vandiver of the University of Texas will be visiting professor at the University of Indiana during the present term.

Assistant Professor E. L. Welker of the University of Illinois has been promoted to an associate professorship.

Assistant Professor L. B. Williams of Hamilton College has been appointed to an assistant professorship at Reed College.

The following appointments to instructorships are announced:

Catholic University of America: R. W. Moller, S. J. Rosenfeld

Hunter College: Paul Brock

Iowa State College: Mrs. R. S. Banton, Mrs. J. V. Carr, Bette L. Flatland, J. W. Markey, J. H. Watson

Ohio State University: Dr. Marjorie Alden

United States Naval Academy: B. H. Buikstra

University of Buffalo: Helen M. Mazzuca, L. O. Ramer

University of Illinois: W. A. Ferguson, J. E. Schubert, B. E. Meserve, Vivian R. Nuess

University of Missouri: Dr. P. B. Burcham

University of Oregon: Mrs. Ethel Lawrence, W. G. Scobert

Professor H. M. Ackley of Western Michigan College died February 8, 1947.

H. G. Avers of the United States Coast and Geodetic Survey died January 19, 1947.

Professor B. F. Finkel of Drury College died February 5, 1947. An account of his life and many services to the Association will appear in a later issue of this MONTHLY.

Professor J. V. Uspensky of Stanford University died January 27, 1947.

Reverend G. L. Winkelmann of St. John's University, Collegeville, Minnesota, died January 23, 1947.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.
Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS, Wichita, April 19, 1947
KENTUCKY, Lexington, May 10, 1947
LOUISIANA-MISSISSIPPI, Hattiesburg, Mississippi, April 25–26, 1947
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May 3, 1947
METROPOLITAN NEW YORK, Brooklyn, April 19, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C, April 18–19, 1947
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS, Lubbock, April 25–26, 1947
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

REPORT OF THE TREASURER FOR THE YEAR 1946

The following report of the Secretary-Treasurer as Treasurer for the year, 1946, has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I: TOTAL FUNDS OF THE ASSOCIATION ON DECEMBER 31, 1945

(See Treasurer's report, pp. 237–240 of the MONTHLY for April, 1946)

Current Fund (checking account)		$ 3,347.26
Savings Account (Ithaca Savings Bank)		1,007.50
Savings Account (OberlinSavings Bank)		648.96
Invested Funds (Cleveland Trust Company)		
Carus Fund	$ 9,455.75	
Chace Fund	9,733.84	
Houck Fund	9,279.29	
Chauvenet Fund	658.85	
Life Membership Fund	718.09	
General Fund	27,547.10	57,392.92
		$62,396.64

II. Current Fund Account for 1946

Receipts		Expenditures	
Balance, Jan. 1, 1946	$ 3,347.26	Monthly	
Dues	9,786.02	Publication	$ 6,029.46
Initiation fees	442.00	Reprints	227.92
Subscriptions	3,139.45	Editor-in-Chief's office	458.68
Sale of back numbers Monthly	295.27	Secretary-Treasurer's office	
Advertisements	726.00	Clerical help	3,502.35
Interest on General Fund	796.87	Postage	414.95
Interest on Carus Fund	273.61	Printing	305.01
Interest on Chace Fund	281.65	Office supplies	220.55
Interest on Houck Fund	268.49	Bank fee	100.00
Interest on Chauvenet Fund	19.08	Exec. and Finance Committees	477.35
Interest on Life Membership Fund	20.78	Regional Governors	200.76
Interest from Hardy Fund	120.00	Sections	213.43
Sale of Archibald's Outline	231.40	Subventions	
Sale of monographs (Carus)	952.75	Amer. Math. Society	100.00
Sale of Papyrus (Chace)	158.00	Mathematical Reviews	350.00
Sale of exchange periodicals	40.00	Back numbers Monthly	114.60
From Oberlin Savings Bank	27.04	Meetings	21.09
Miscellaneous sources	2.28	Coordinating Committee	121.80
Transferred from General Fund	1,974.36	Coop. Committee on Sci. Teaching	106.82
		Amer. Council on Education	215.36
		To General Fund	500.00
		Subscriptions to Annals	15.00
		B. F. Finkel (Hardy Fund)	120.00
		Register	753.85
		Bank charges	
		Exchange fees	9.55
		Bad checks	.50
		Transferred	
		To Carus Fund	1,226.36
		To Chace Fund	439.65
		To Houck Fund	268.49
		To Chauvenet Fund	19.08
		To Life Membership Fund	20.78
		Balance, Dec. 31, 1946	6,348.92
	$22,902.31		$22,902.31

III. Savings Account, Ithaca Savings Bank

Balance, Jan. 1, 1946	$1,007.50	Balance, Dec. 31, 1946	$1,022.66
Interest	15.16		
	$1,022.66		$1,022.66

IV. Savings Account, Oberlin Savings Bank

Balance, Jan. 1, 1946	$ 648.96	Final Payment	$ 27.04
		Final loss on the account	621.92
	$ 648.96		$ 648.96

V. Invested Funds, Cleveland Trust Company

Cash Balance, Jan. 1, 1946	$ 279.92	Decrease in value of securities	$ 1,022.62
Market value of securities, Dec. 31, 1945	57,113.00	Market value of securities, Dec. 31, 1946	56,696.00
From Current Fund	500.00	Cash balance, Dec. 31, 1946	174.30
	$57,892.92		$57,892.92

List of Securities

	Par Value	Market Value Dec. 31, 1946
U. S. Savings Bonds, 1947	$ 700.00	$ 686.00
U. S. Treasury Bonds, 2%, 1950	3,000.00	3,030.00
U. S. Treasury Bonds, 2%, 1954	2,000.00	2,060.00
U. S. Treasury Bonds, 2½%, 1969	2,000.00	2,080.00
U. S. Treasury Bonds, 2½%, 1972	1,000.00	1,030.00
U. S. Treasury Bonds, 2¼%, 1962	5,000.00	5,100.00
U. S. Treasury Bonds, 1¾%, 1948	2,000.00	2,020.00
U. S. Savings Bonds, Ser. G, 2½%, 1953	3,000.00	2,850.00
U. S. Savings Bonds, Ser. G, 2½%, 1954	8,200.00	7,790.00
U. S. Savings Bonds, Ser. G, 2½%, 1958	3,000.00	2,970.00
Canadian Nat. Ry. Co. Bonds, 4½%, 1956	2,000.00	2,360.00
C. and O. Ry. Co. Ref. Bonds, Ser. D, 3¾%, 1996	3,000.00	3,180.00
Columbus and So. Ohio Elec. Co. Bonds, 3¼%, 1970	2,000.00	2,200.00
New York Steam Corp. 1st Mort. Bond, 3½%, 1963	1,000.00	1,060.00
Amer. Tobacco Co. Bonds, 3%, 1969	4,000.00	4,240.00
C. and O. Ry. Co. common stock, 25 shares		1,350.00
Amer. Tel. & Tel. Co. common stock, 30 shares		5,160.00
Standard Oil Co. New Jersey common stock, 20 shares		1,380.00
Atch. Top., Santa Fe R.R. non cum. pfd. stock, 15 sh.		1,590.00
Commonwealth Edison Co. common stock, 80 shares		2,720.00
Dana Corp. cum. pfd. stock, 20 shares		1,840.00
		$56,696.00

VI. Carus Fund

Balance, Jan. 1, 1946	$ 9,455.75	Decrease in value of securities	$ 168.46
Sale of monographs	952.75		
Interest	273.61	Balance, Dec. 31, 1946	10,513.65
	$10,682.11		$10,682.11

VII. Chace Fund

Balance, Jan. 1. 1946	$ 9,733.84	Decrease in value of securities	$ 173.43
Sale of Papyrus	158.00		
Interest	281.65	Balance, Dec. 31, 1946	10,000.06
	$10,173.49		$10,173.49

VIII. Houck Fund

Balance, Jan. 1, 1946	$9,279.29	Decrease in value of securities	$ 165.15
Interest	268.49	Balance, Dec. 31, 1946	9,382.63
	$9,547.78		$9,547.78

IX. Chauvenet Fund

Balance, Jan. 1, 1946	$ 658.85	Decrease in value of securities	$ 12.35
Interest	19.08	Balance, Dec. 31, 1946	665.58
	$ 677.93		$ 677.93

X. Life Membership Fund

Balance, Jan. 1, 1946	$ 718.09	To General Fund	$ 76.82
Interest	20.78	Decrease in value of securities	13.45
		Balance, Dec. 31, 1946	648.60
	$ 738.87		$ 738.87

XI. General Fund

Balance, Jan. 1, 1946	$27,547.10	Decrease in value of securities	$ 489.78
From Current Fund	500.00	Transferred to Current Fund	1,974.36
From Life Membership Fund	76.82	Balance, Dec. 31, 1946	25,659.78
	$28,123.92		$28,123.92

XII. Total Funds of the Association, December 31, 1946

Current Fund (checking account)		$ 6,348.92
Savings Account (Ithaca Savings Bank)		1,022.66
Invested Funds (Cleveland Trust Company)		
Carus Fund	$10,513.65	
Chace Fund	10,000.06	
Houck Fund	9,382.63	
Chauvenet Fund	665.58	
Life Membership Fund	648.60	
General Fund	25,659.78	$56,870.30
		$64,241.88

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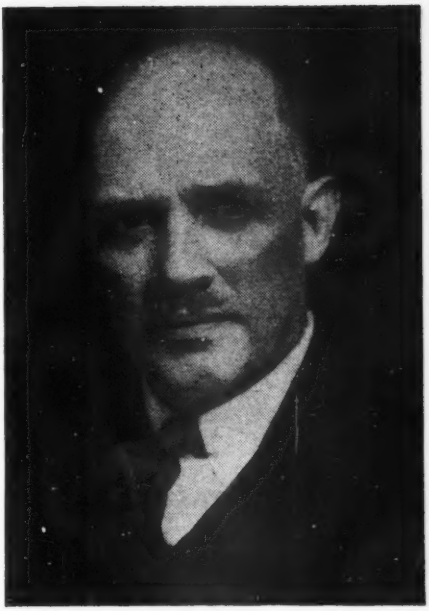

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